

# Computer algebra independent integration tests

4-Trig-functions/4.6-Cosecant/4.6.3.1-a+b-csc<sup>m</sup>-d-csc<sup>n</sup>-A+B-csc-

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# Chapter 1

## Introduction

This report gives the result of running the computer algebra independent integration problems. The listing of the problems are maintained by and can be downloaded from <https://rulebasedintegration.org>

The number of integrals in this report is [ 24 ]. This is test number [ 132 ].

### 1.1 Listing of CAS systems tested

The following systems were tested at this time.

1. Mathematica 12.1 (64 bit) on windows 10.
2. Rubi 4.16.1 in Mathematica 12 on windows 10.
3. Maple 2020 (64 bit) on windows 10.
4. Maxima 5.43 on Linux. (via sagemath 8.9)
5. Fricas 1.3.6 on Linux (via sagemath 9.0)
6. Sympy 1.5 under Python 3.7.3 using Anaconda distribution.
7. Giac/Xcas 1.5 on Linux. (via sagemath 8.9)

Maxima, Fricas and Giac/Xcas were called from inside SageMath. This was done using SageMath integrate command by changing the name of the algorithm to use the different CAS systems.

Sympy was called directly using Python.

## 1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or Hypergeometric2F1 functions. RootSum and RootOf are not allowed.

If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

System	solved	Failed
Rubi	% 100. ( 24 )	% 0. ( 0 )
Mathematica	% 100. ( 24 )	% 0. ( 0 )
Maple	% 100. ( 24 )	% 0. ( 0 )
Maxima	% 100. ( 24 )	% 0. ( 0 )
Fricas	% 100. ( 24 )	% 0. ( 0 )
Sympy	% 37.5 ( 9 )	% 62.5 ( 15 )
Giac	% 100. ( 24 )	% 0. ( 0 )

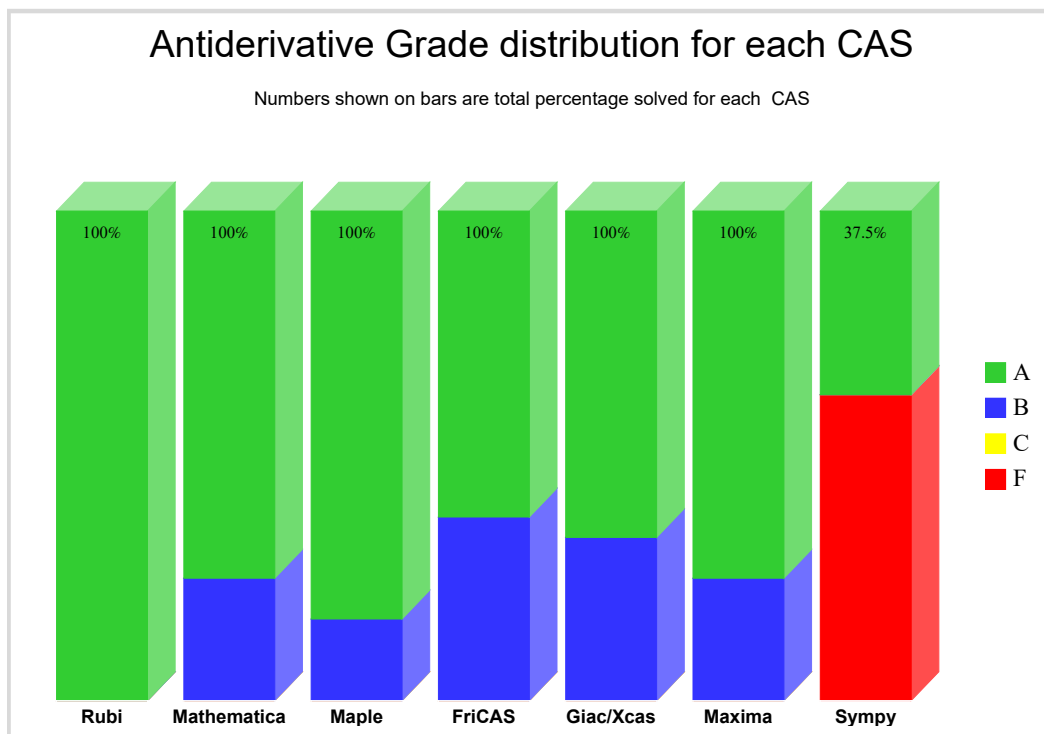
The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

grade	description
A	Integral was solved and antiderivative is optimal in quality and leaf size.
B	Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size.
C	Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons <ol style="list-style-type: none"> <li>1. antiderivative contains a hypergeometric function and the optimal antiderivative does not.</li> <li>2. antiderivative contains a special function and the optimal antiderivative does not.</li> <li>3. antiderivative contains the imaginary unit and the optimal antiderivative does not.</li> </ol>
F	Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised.

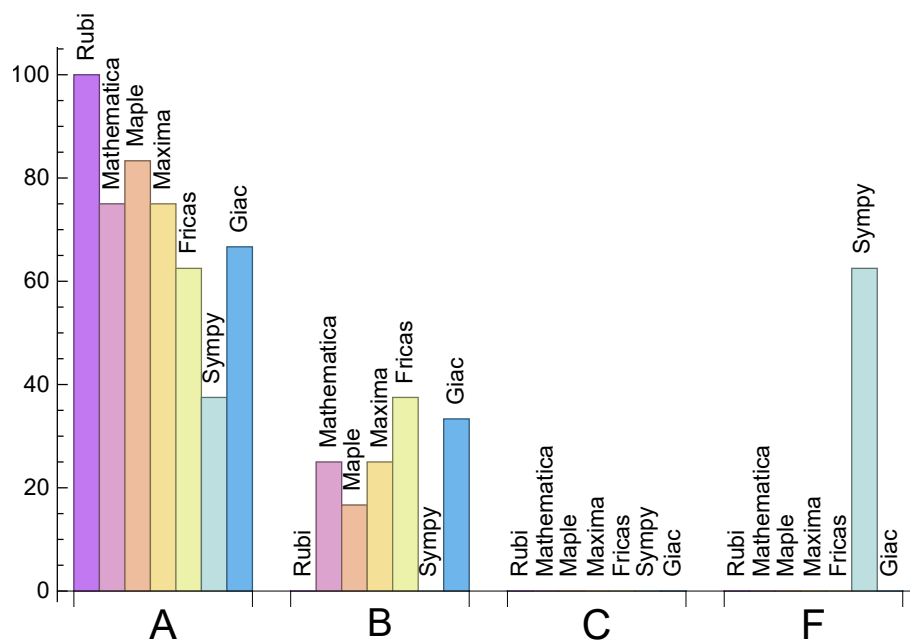
Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

System	% A grade	% B grade	% C grade	% F grade
Rubi	100.	0.	0.	0.
Mathematica	75.	25.	0.	0.
Maple	83.33	16.67	0.	0.
Maxima	75.	25.	0.	0.
Fricas	62.5	37.5	0.	0.
Sympy	37.5	0.	0.	62.5
Giac	66.67	33.33	0.	0.

The following is a Bar chart illustration of the data in the above table.



The figure below compares the CAS systems for each grade level.



### 1.3 Performance

The table below summarizes the performance of each CAS system in terms of CPU time and leaf size of results.

System	Mean time (sec)	Mean size	Normalized mean	Median size	Normalized median
Rubi	0.07	44.25	1.	40.	1.
Mathematica	0.04	72.21	1.54	59.	1.68
Maple	0.04	54.5	1.37	49.5	1.16
Maxima	1.02	90.5	2.14	74.5	2.09
Fricas	0.49	194.62	4.19	139.5	4.86
Sympy	9.25	81.22	2.93	82.	2.78
Giac	1.41	108.71	2.52	116.5	2.37

## 1.4 list of integrals that has no closed form antiderivative

{}

## 1.5 list of integrals solved by CAS but has no known antiderivative

**Rubi** {}

**Mathematica** {}

**Maple** {}

**Maxima** {}

**Fricas** {}

**Sympy** {}

**Giac** {}

## 1.6 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not mean necessarily that the anti-derivative is wrong, as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it easier to do further investigation to determine why it was not possible to verify the result produced.

**Rubi** {}

**Mathematica** {}

**Maple** Verification phase not implemented yet.

**Maxima** Verification phase not implemented yet.

**Fricas** Verification phase not implemented yet.

**Sympy** Verification phase not implemented yet.

**Giac** Verification phase not implemented yet.



## 1.7 Timing

The command `AboluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of _int',int(expr,x)),output='realtime')
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call has completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out is not counted in the final statistics.

## 1.8 Verification

A verification phase was applied on the result of integration for Rubi and Mathematica. Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative produced was correct.

Verification phase has 3 minutes time out. An integral whose result was not verified could still be correct. Further investigation is needed on those integrals which failed verifications. Such integrals are marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

## 1.9 Important notes about some of the results

### 1.9.1 Important note about Maxima results

Since these integrals are run in a batch mode, using an automated script, and by using `sagemath` (SageMath uses Maxima), then any integral where Maxima needs an interactive response from the user to answer a question during evaluation of the integral in order to complete the integration, will fail and is counted as failed.

The exception raised is `ValueError`. Therefore Maxima result below is lower than what could result if Maxima was run directly and each question Maxima asks was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the Timofeev test file, there were about 30 such integrals out of total 705, or about 4 percent. This percentage can be higher or lower depending on the specific input test file.

Such integrals can be identified by looking at the output of the integration in each section for Maxima. If the output was an exception `ValueError` then this is most likely due to this reason.

Maxima integrate was run using SageMath with the following settings set by default

```
'besselexpand : true'
'display2d : false'
'domain : complex'
'keepfloat : true'
'load(to_poly_solve)'
'load(simplify_sum)'
'load(abs_integrate)' 'load(diag)'
```

SageMath loading of Maxima `abs_integrate` was found to cause some problem. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib
maxima_lib.set('extra_definite_integration_methods', '[]')
maxima_lib.set('extra_integration_methods', '[]')
```

See <https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/> for reference.

## 1.9.2 Important note about FriCAS and Giac/X-CAS results

There are Few integrals which failed due to SageMath not able to translate the result back to SageMath syntax and not because these CAS system were not able to do the integrations.

These will fail With error `Exception raised: NotImplementedError`

The number of such cases seems to be very small. About 1 or 2 percent of all integrals.

Hopefully the next version of SageMath will have complete translation of FriCAS and XCAS syntax and I will re-run all the tests again when this happens.

## 1.9.3 Important note about finding leaf size of antiderivative

For Mathematica, Rubi and Maple, the builtin system function `LeafSize` is used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special builtin function for this purpose at this time. Therefore the leaf size is determined as follows.

For Fricas, Giac and Maxima (all called via `sagemath`) the following code is used

#see <https://stackoverflow.com/questions/25202346/how-to-obtain-leaf-count-expression-size-in>

```
def tree(expr):
    if expr.operator() is None:
        return expr
    else:
        return [expr.operator()+map(tree, expr.operands())

try:
    # 1.35 is a fudge factor since this estimate of leaf count is bit lower than
    #what it should be compared to Mathematica's
    leafCount = round(1.35*len(flatten(tree(anti))))
except Exception as ee:
    leafCount =1
```

For Sympy, called directly from Python, the following code is used

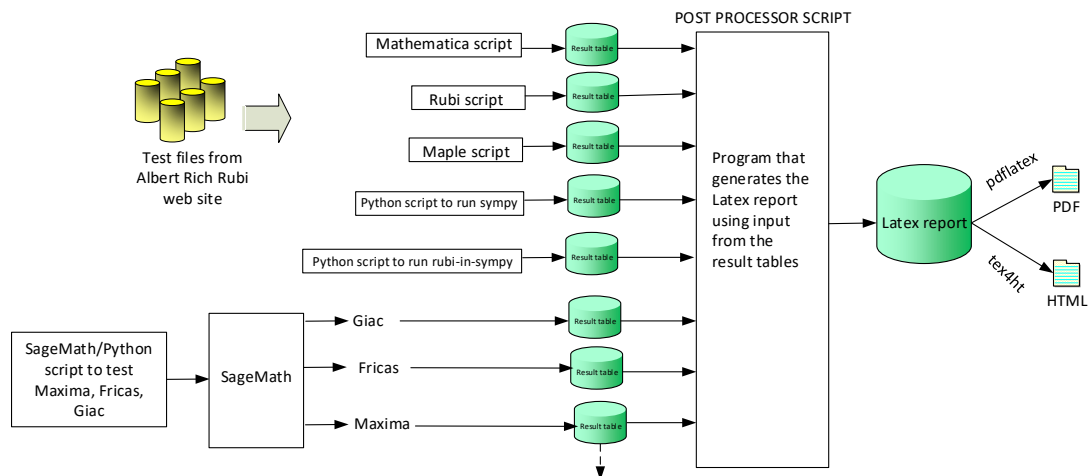
```
try:
    # 1.7 is a fudge factor since it is low side from actual leaf count
    leafCount = round(1.7*count_ops(anti))

except Exception as ee:
    leafCount =1
```

When these cas systems have a builtin function to find the leaf size of expressions, it will be used instead, and these tests run again.

## 1.10 Design of the test system

The following diagram gives a high level view of the current test build system.



One record (line) per one integral result. The line is CSV comma separated. It contains 13 fields. This is description of each record (line)

1. integer, the problem number.
2. integer. 0 or 1 for failed or passed. (this is not the grade field)
3. integer. Leaf size of result.
4. integer. Leaf size of the optimal antiderivative.
5. number. CPU time used to solve this integral. 0 if failed.
6. string. The integral in Latex format
7. string. The input used in CAS own syntax.
8. string. The result (antiderivative) produced by CAS in Latex format
9. string. The optimal antiderivative in Latex format.
10. integer. 0 or 1. Indicates if problem has known antiderivative or not
11. String. The result (antiderivative) in CAS own syntax.
12. String. The grade of the antiderivative. Can be "A", "B", "C", or "F"
13. String. The optimal antiderivative in CAS own syntax.

### High level overview of the CAS independent integration test build system

# Chapter 2

## detailed summary tables of results

### 2.1 List of integrals sorted by grade for each CAS

#### 2.1.1 Rubi

A grade: { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24 }

B grade: { }

C grade: { }

F grade: { }

#### 2.1.2 Mathematica

A grade: { 1, 2, 5, 6, 7, 8, 10, 11, 12, 13, 14, 16, 17, 18, 19, 20, 23, 24 }

B grade: { 3, 4, 9, 15, 21, 22 }

C grade: { }

F grade: { }

#### 2.1.3 Maple

A grade: { 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 13, 15, 16, 17, 19, 20, 21, 22, 23, 24 }

B grade: { 8, 12, 14, 18 }

C grade: { }

F grade: { }

## 2.1.4 Maxima

A grade: { 1, 2, 3, 4, 5, 6, 9, 10, 11, 15, 16, 17, 19, 20, 21, 22, 23, 24 }

B grade: { 7, 8, 12, 13, 14, 18 }

C grade: { }

F grade: { }

## 2.1.5 FriCAS

A grade: { 1, 2, 4, 5, 6, 10, 11, 12, 16, 17, 18, 19, 22, 23, 24 }

B grade: { 3, 7, 8, 9, 13, 14, 15, 20, 21 }

C grade: { }

F grade: { }

## 2.1.6 Sympy

A grade: { 8, 11, 14, 16, 17, 18, 22, 23, 24 }

B grade: { }

C grade: { }

F grade: { 1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 13, 15, 19, 20, 21 }

## 2.1.7 Giac

A grade: { 1, 2, 3, 4, 6, 7, 8, 11, 13, 14, 17, 19, 20, 21, 22, 24 }

B grade: { 5, 9, 10, 12, 15, 16, 18, 23 }

C grade: { }

F grade: { }

## 2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by table below. The elapsed time is in seconds. For failed result it is given as F(-1) if the failure was due to timeout. It is given as F(-2) if the failure was due to an exception being raised, which could indicate a bug in the

system. If the failure was due to integral not being evaluated within the time limit, then it is given just an F.

In this table, the column **normalized size** is defined as  $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$

Problem 1	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	A	F	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	91	91	163	99	196	439	0	209
normalized size	1	1.	1.79	1.09	2.15	4.82	0.	2.3
time (sec)	N/A	0.102	0.057	0.051	0.984	0.502	0.	1.446

Problem 2	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	A	F	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	72	72	116	77	117	363	0	166
normalized size	1	1.	1.61	1.07	1.62	5.04	0.	2.31
time (sec)	N/A	0.096	0.045	0.043	0.988	0.5	0.	1.357

Problem 3	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	B	A	A	B	F	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	51	51	137	57	108	273	0	126
normalized size	1	1.	2.69	1.12	2.12	5.35	0.	2.47
time (sec)	N/A	0.058	0.04	0.045	0.999	0.493	0.	1.368

Problem 4	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	B	A	A	A	F	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	33	33	72	50	68	147	0	63
normalized size	1	1.	2.18	1.52	2.06	4.45	0.	1.91
time (sec)	N/A	0.059	0.023	0.079	0.989	0.508	0.	1.356

Problem 5	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	A	F	B
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	42	42	33	49	63	97	0	107
normalized size	1	1.	0.79	1.17	1.5	2.31	0.	2.55
time (sec)	N/A	0.073	0.063	0.078	0.99	0.476	0.	1.497

Problem 6	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	A	F	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	54	54	43	62	81	128	0	128
normalized size	1	1.	0.8	1.15	1.5	2.37	0.	2.37
time (sec)	N/A	0.094	0.084	0.089	0.962	0.504	0.	1.391

Problem 7	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	B	B	F	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	61	61	117	65	162	344	0	144
normalized size	1	1.	1.92	1.07	2.66	5.64	0.	2.36
time (sec)	N/A	0.092	0.045	0.026	1.014	0.497	0.	1.381

Problem 8	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	B	B	B	A	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	17	17	17	38	57	85	54	20
normalized size	1	1.	1.	2.24	3.35	5.	3.18	1.18
time (sec)	N/A	0.072	0.016	0.02	1.057	0.456	2.153	1.367

Problem 9	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	B	A	A	B	F	B
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	38	38	79	44	92	223	0	136
normalized size	1	1.	2.08	1.16	2.42	5.87	0.	3.58
time (sec)	N/A	0.051	0.035	0.023	1.116	0.496	0.	1.333



Problem 10	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	A	F	B
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	27	27	46	38	54	132	0	81
normalized size	1	1.	1.7	1.41	2.	4.89	0.	3.
time (sec)	N/A	0.053	0.024	0.041	1.112	0.501	0.	1.366

Problem 11	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	A	A	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	29	29	25	40	50	68	54	47
normalized size	1	1.	0.86	1.38	1.72	2.34	1.86	1.62
time (sec)	N/A	0.055	0.022	0.036	1.04	0.476	13.381	1.259

Problem 12	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	B	B	A	F	B
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	17	17	17	35	49	35	0	76
normalized size	1	1.	1.	2.06	2.88	2.06	0.	4.47
time (sec)	N/A	0.061	0.007	0.036	1.077	0.471	0.	1.359

Problem 13	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	B	B	F	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	61	61	117	65	162	344	0	144
normalized size	1	1.	1.92	1.07	2.66	5.64	0.	2.36
time (sec)	N/A	0.09	0.038	0.027	1.	0.497	0.	1.426

Problem 14	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	B	B	B	A	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	17	17	17	38	57	85	54	20
normalized size	1	1.	1.	2.24	3.35	5.	3.18	1.18
time (sec)	N/A	0.071	0.012	0.021	1.022	0.456	2.146	1.385

Problem 15	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	B	A	A	B	F	B
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	38	38	79	44	92	223	0	136
normalized size	1	1.	2.08	1.16	2.42	5.87	0.	3.58
time (sec)	N/A	0.049	0.035	0.025	1.001	0.496	0.	1.413

Problem 16	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	27	27	46	38	45	132	75	81
normalized size	1	1.	1.7	1.41	1.67	4.89	2.78	3.
time (sec)	N/A	0.052	0.022	0.023	1.015	0.507	11.496	1.393

Problem 17	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	A	A	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	29	29	25	40	50	68	85	47
normalized size	1	1.	0.86	1.38	1.72	2.34	2.93	1.62
time (sec)	N/A	0.055	0.011	0.021	0.994	0.479	6.966	1.453

Problem 18	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	B	B	A	A	B
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	17	17	17	35	49	35	83	76
normalized size	1	1.	1.	2.06	2.88	2.06	4.88	4.47
time (sec)	N/A	0.061	0.006	0.028	0.981	0.472	10.56	1.315

Problem 19	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	A	F	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	91	91	163	99	196	439	0	209
normalized size	1	1.	1.79	1.09	2.15	4.82	0.	2.3
time (sec)	N/A	0.096	0.057	0.049	0.988	0.504	0.	1.479

Problem 20	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	B	F	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	70	70	116	77	117	363	0	166
normalized size	1	1.	1.66	1.1	1.67	5.19	0.	2.37
time (sec)	N/A	0.09	0.043	0.045	1.05	0.505	0.	1.647

Problem 21	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	B	A	A	B	F	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	51	51	137	57	108	274	0	126
normalized size	1	1.	2.69	1.12	2.12	5.37	0.	2.47
time (sec)	N/A	0.055	0.035	0.048	1.004	0.496	0.	1.498

Problem 22	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	B	A	A	A	A	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	33	33	72	50	55	149	82	66
normalized size	1	1.	2.18	1.52	1.67	4.52	2.48	2.
time (sec)	N/A	0.058	0.027	0.041	0.995	0.503	12.314	1.372

Problem 23	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	A	A	B
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	42	42	35	49	63	97	105	107
normalized size	1	1.	0.83	1.17	1.5	2.31	2.5	2.55
time (sec)	N/A	0.069	0.055	0.041	0.994	0.473	10.545	1.434

Problem 24	Optimal	Rubi	Mathematica	Maple	Maxima	Fricas	Sympy	Giac
grade	A	A	A	A	A	A	A	A
verified	N/A	Yes	Yes	TBD	TBD	TBD	TBD	TBD
size	54	54	44	62	81	128	139	128
normalized size	1	1.	0.81	1.15	1.5	2.37	2.57	2.37
time (sec)	N/A	0.09	0.11	0.048	1.007	0.482	13.726	1.527

## 2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio  $\frac{\text{number of rules}}{\text{integrand size}}$  is given. The larger this ratio is, the harder the integral was to solve. In this test, problem number [2] had the largest ratio of [ 0.2414 ]

Table 2.1: Rubi specific breakdown of results for each integral

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
1	A	7	6	1.	29	0.207
2	A	7	7	1.	29	0.241
3	A	6	6	1.	27	0.222
4	A	5	5	1.	27	0.185
5	A	5	5	1.	29	0.172
6	A	7	6	1.	29	0.207
7	A	4	4	1.	30	0.133
8	A	3	3	1.	30	0.1
9	A	3	3	1.	28	0.107
10	A	4	4	1.	28	0.143
11	A	3	3	1.	30	0.1
12	A	3	3	1.	30	0.1
13	A	4	4	1.	30	0.133
14	A	3	3	1.	30	0.1
15	A	3	3	1.	28	0.107
16	A	4	4	1.	28	0.143
17	A	3	3	1.	30	0.1
18	A	3	3	1.	30	0.1
19	A	7	6	1.	31	0.194
20	A	7	7	1.	31	0.226
21	A	6	6	1.	29	0.207
22	A	5	5	1.	29	0.172
23	A	5	5	1.	31	0.161

Continued on next page

Table 2.1 – continued from previous page

#	grade	number of steps used	number of unique rules	normalized antiderivative leaf size	integrand leaf size	$\frac{\text{number of rules}}{\text{integrand leaf size}}$
24	A	7	6	1.	31	0.194



# Chapter 3

## Listing of integrals

### 3.1 $\int \csc^3(c+dx)(a+a \csc(c+dx))(A+A \csc(c+dx)) dx$

**Optimal.** Leaf size=91

$$\frac{2aA \cot^3(c+dx)}{3d} - \frac{2aA \cot(c+dx)}{d} - \frac{7aA \tanh^{-1}(\cos(c+dx))}{8d} - \frac{aA \cot(c+dx) \csc^3(c+dx)}{4d} - \frac{7aA \cot(c+dx) \csc^3(c+dx)}{8d}$$

[Out]  $(-7*a*A*ArcTanh[Cos[c + d*x]])/(8*d) - (2*a*A*Cot[c + d*x])/d - (2*a*A*Cot[c + d*x]^3)/(3*d) - (7*a*A*Cot[c + d*x]*Csc[c + d*x])/(8*d) - (a*A*Cot[c + d*x]*Csc[c + d*x]^3)/(4*d)$

---

**Rubi [A]** time = 0.102302, antiderivative size = 91, normalized size of antiderivative = 1., number of steps used = 7, number of rules used = 6, integrand size = 29,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.207$ , Rules used = {21, 3788, 3767, 4046, 3768, 3770}

$$\frac{2aA \cot^3(c+dx)}{3d} - \frac{2aA \cot(c+dx)}{d} - \frac{7aA \tanh^{-1}(\cos(c+dx))}{8d} - \frac{aA \cot(c+dx) \csc^3(c+dx)}{4d} - \frac{7aA \cot(c+dx) \csc^3(c+dx)}{8d}$$

Antiderivative was successfully verified.

[In]  $\text{Int}[Csc[c + d*x]^3*(a + a*Csc[c + d*x])*(A + A*Csc[c + d*x]), x]$

[Out]  $(-7*a*A*ArcTanh[Cos[c + d*x]])/(8*d) - (2*a*A*Cot[c + d*x])/d - (2*a*A*Cot[c + d*x]^3)/(3*d) - (7*a*A*Cot[c + d*x]*Csc[c + d*x])/(8*d) - (a*A*Cot[c + d*x]*Csc[c + d*x]^3)/(4*d)$

Rule 21

```
Int[(u_.)*((a_) + (b_.)*(v_))^(m_.)*((c_) + (d_.)*(v_))^(n_.), x_Symbol] :=
  Dist[(b/d)^m, Int[u*(c + d*v)^(m + n), x], x] /; FreeQ[{a, b, c, d, n}, x]
  && EqQ[b*c - a*d, 0] && IntegerQ[m] && (!IntegerQ[n] || SimplerQ[c + d*x,
  a + b*x])
```

### Rule 3788

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_.))^2, x_Symbol] := Dist[(2*a*b)/d, Int[(d*Csc[e + f*x])^(n + 1), x], x]
+ Int[(d*Csc[e + f*x])^n*(a^2 + b^2*Csc[e + f*x]^2), x] /; FreeQ[{a, b, d,
e, f, n}, x]
```

### Rule 3767

```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] := -Dist[d^(-1), Subst[Int[Expa
ndIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c,
d}, x] && IGtQ[n/2, 0]
```

### Rule 4046

```
Int[(csc[(e_.) + (f_.)*(x_)]*(b_.))^(m_.)*(csc[(e_.) + (f_.)*(x_)]^2*(C_.)
+ (A_.)), x_Symbol] := -Simp[(C*Cot[e + f*x]*(b*Csc[e + f*x])^m)/(f*(m + 1))
, x] + Dist[(C*m + A*(m + 1))/(m + 1), Int[(b*Csc[e + f*x])^m, x], x] /; Fr
eeQ[{b, e, f, A, C, m}, x] && NeQ[C*m + A*(m + 1), 0] && !LeQ[m, -1]
```

### Rule 3768

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := -Simp[(b*Cos[c + d*x]
)*(b*Csc[c + d*x])^(n - 1))/(d*(n - 1)), x] + Dist[(b^2*(n - 2))/(n - 1), I
nt[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] &&
IntegerQ[2*n]
```

### Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] := -Simp[ArcTanh[Cos[c + d*x]]/d, x]
/; FreeQ[{c, d}, x]
```

### Rubi steps



$$\begin{aligned}
\int \csc^3(c+dx)(a+a\csc(c+dx))(A+A\csc(c+dx))dx &= \frac{A \int \csc^3(c+dx)(a+a\csc(c+dx))^2 dx}{a} \\
&= \frac{A \int \csc^3(c+dx)(a^2+a^2\csc^2(c+dx)) dx}{a} + (2aA) \int \csc^4(c+dx) dx \\
&= -\frac{aA \cot(c+dx) \csc^3(c+dx)}{4d} + \frac{1}{4}(7aA) \int \csc^3(c+dx) dx \\
&= -\frac{2aA \cot(c+dx)}{d} - \frac{2aA \cot^3(c+dx)}{3d} - \frac{7aA \cot(c+dx) \csc^3(c+dx)}{8d} \\
&= -\frac{7aA \tanh^{-1}(\cos(c+dx))}{8d} - \frac{2aA \cot(c+dx)}{d} - \frac{2aA \cot^3(c+dx)}{3d}
\end{aligned}$$

**Mathematica [A]** time = 0.0568929, size = 163, normalized size = 1.79

$$-\frac{4aA \cot(c+dx)}{3d} - \frac{aA \csc^4\left(\frac{1}{2}(c+dx)\right)}{64d} - \frac{7aA \csc^2\left(\frac{1}{2}(c+dx)\right)}{32d} + \frac{aA \sec^4\left(\frac{1}{2}(c+dx)\right)}{64d} + \frac{7aA \sec^2\left(\frac{1}{2}(c+dx)\right)}{32d} + \frac{7aA \cot(c+dx)}{8d}$$

Antiderivative was successfully verified.

[In] Integrate[Csc[c + d\*x]^3\*(a + a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x]),x]

[Out]  $(-4*a*A*\text{Cot}[c + d*x])/(3*d) - (7*a*A*\text{Csc}[(c + d*x)/2]^2)/(32*d) - (a*A*\text{Csc}[(c + d*x)/2]^4)/(64*d) - (2*a*A*\text{Cot}[c + d*x]*\text{Csc}[c + d*x]^2)/(3*d) - (7*a*A*\text{Log}[\text{Cos}[(c + d*x)/2]])/(8*d) + (7*a*A*\text{Log}[\text{Sin}[(c + d*x)/2]])/(8*d) + (7*a*A*\text{Sec}[(c + d*x)/2]^2)/(32*d) + (a*A*\text{Sec}[(c + d*x)/2]^4)/(64*d)$

**Maple [A]** time = 0.051, size = 99, normalized size = 1.1

$$-\frac{7Aa \cot(dx+c) \csc(dx+c)}{8d} + \frac{7Aa \ln(\csc(dx+c) - \cot(dx+c))}{8d} - \frac{4Aa \cot(dx+c)}{3d} - \frac{2Aa \cot(dx+c) (\csc(dx+c))^2}{3d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int(csc(d\*x+c)^3\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x)

[Out]  $-7/8*a*A*\cot(d*x+c)*\csc(d*x+c)/d+7/8/d*A*a*\ln(\csc(d*x+c)-\cot(d*x+c))-4/3*a*A*\cot(d*x+c)/d-2/3*a*A*\cot(d*x+c)*\csc(d*x+c)^2/d-1/4*a*A*\cot(d*x+c)*\csc(d*x+c)^3/d$

---

**Maxima [A]** time = 0.984284, size = 196, normalized size = 2.15

$$\frac{3 Aa \left( \frac{2(3 \cos(dx+c)^3 - 5 \cos(dx+c))}{\cos(dx+c)^4 - 2 \cos(dx+c)^2 + 1} - 3 \log(\cos(dx+c) + 1) + 3 \log(\cos(dx+c) - 1) \right) + 12 Aa \left( \frac{2 \cos(dx+c)}{\cos(dx+c)^2 - 1} - \log(\cos(dx+c) + 1) + \log(\cos(dx+c) - 1) \right)}{48 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^3\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="maxima")

[Out] 1/48\*(3\*A\*a\*(2\*(3\*cos(d\*x + c)^3 - 5\*cos(d\*x + c))/(cos(d\*x + c)^4 - 2\*cos(d\*x + c)^2 + 1) - 3\*log(cos(d\*x + c) + 1) + 3\*log(cos(d\*x + c) - 1)) + 12\*A\*a\*(2\*cos(d\*x + c)/(cos(d\*x + c)^2 - 1) - log(cos(d\*x + c) + 1) + log(cos(d\*x + c) - 1)) - 32\*(3\*tan(d\*x + c)^2 + 1)\*A\*a/tan(d\*x + c)^3)/d

---

**Fricas [A]** time = 0.501738, size = 439, normalized size = 4.82

$$\frac{42 Aa \cos(dx+c)^3 - 54 Aa \cos(dx+c) - 21 (Aa \cos(dx+c)^4 - 2 Aa \cos(dx+c)^2 + Aa) \log\left(\frac{1}{2} \cos(dx+c) + \frac{1}{2}\right) + 21 (Aa \cos(dx+c)^4 - 2 Aa \cos(dx+c)^2 + Aa) \log\left(\frac{1}{2} \cos(dx+c) - \frac{1}{2}\right)}{48 (d \cos(dx+c))}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^3\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="fricas")

[Out] 1/48\*(42\*A\*a\*cos(d\*x + c)^3 - 54\*A\*a\*cos(d\*x + c) - 21\*(A\*a\*cos(d\*x + c)^4 - 2\*A\*a\*cos(d\*x + c)^2 + A\*a)\*log(1/2\*cos(d\*x + c) + 1/2) + 21\*(A\*a\*cos(d\*x + c)^4 - 2\*A\*a\*cos(d\*x + c)^2 + A\*a)\*log(-1/2\*cos(d\*x + c) + 1/2) + 32\*(2\*A\*a\*cos(d\*x + c)^3 - 3\*A\*a\*cos(d\*x + c))\*sin(d\*x + c)/(d\*cos(d\*x + c)^4 - 2\*d\*cos(d\*x + c)^2 + d)

---

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$Aa \left( \int \csc^3(c + dx) dx + \int 2 \csc^4(c + dx) dx + \int \csc^5(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)**3*(a+a*csc(d*x+c))*(A+A*csc(d*x+c)),x)
```

```
[Out] A*a*(Integral(csc(c + d*x)**3, x) + Integral(2*csc(c + d*x)**4, x) + Integr
al(csc(c + d*x)**5, x))
```

**Giac [A]** time = 1.44568, size = 209, normalized size = 2.3

$$3 A a \tan\left(\frac{1}{2} d x + \frac{1}{2} c\right)^4 + 16 A a \tan\left(\frac{1}{2} d x + \frac{1}{2} c\right)^3 + 48 A a \tan\left(\frac{1}{2} d x + \frac{1}{2} c\right)^2 + 168 A a \log\left(\left|\tan\left(\frac{1}{2} d x + \frac{1}{2} c\right)\right|\right) + 144 A a$$

192 d

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^3*(a+a*csc(d*x+c))*(A+A*csc(d*x+c)),x, algorithm="giac
")
```

```
[Out] 1/192*(3*A*a*tan(1/2*d*x + 1/2*c)^4 + 16*A*a*tan(1/2*d*x + 1/2*c)^3 + 48*A*
a*tan(1/2*d*x + 1/2*c)^2 + 168*A*a*log(abs(tan(1/2*d*x + 1/2*c)))) + 144*A*a
*tan(1/2*d*x + 1/2*c) - (350*A*a*tan(1/2*d*x + 1/2*c)^4 + 144*A*a*tan(1/2*d
*x + 1/2*c)^3 + 48*A*a*tan(1/2*d*x + 1/2*c)^2 + 16*A*a*tan(1/2*d*x + 1/2*c)
+ 3*A*a)/tan(1/2*d*x + 1/2*c)^4/d
```

### 3.2 $\int \csc^2(c+dx)(a+a \csc(c+dx))(A+A \csc(c+dx)) dx$

**Optimal.** Leaf size=72

$$\frac{5aA \cot(c+dx)}{3d} - \frac{aA \tanh^{-1}(\cos(c+dx))}{d} - \frac{aA \cot(c+dx) \csc^2(c+dx)}{3d} - \frac{aA \cot(c+dx) \csc(c+dx)}{d}$$

[Out]  $-\left(\frac{aA \operatorname{ArcTanh}[\cos[c+d*x]]}{d}\right) - \frac{5*a*A*\cot[c+d*x]}{(3*d)} - \frac{a*A*\cot[c+d*x]*\csc[c+d*x]}{d} - \frac{a*A*\cot[c+d*x]*\csc[c+d*x]^2}{(3*d)}$

**Rubi [A]** time = 0.0962889, antiderivative size = 72, normalized size of antiderivative = 1., number of steps used = 7, number of rules used = 7, integrand size = 29,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.241$ , Rules used = {21, 3788, 3768, 3770, 4046, 3767, 8}

$$\frac{5aA \cot(c+dx)}{3d} - \frac{aA \tanh^{-1}(\cos(c+dx))}{d} - \frac{aA \cot(c+dx) \csc^2(c+dx)}{3d} - \frac{aA \cot(c+dx) \csc(c+dx)}{d}$$

Antiderivative was successfully verified.

[In]  $\text{Int}[\text{Csc}[c+d*x]^2*(a+a*\text{Csc}[c+d*x])*(A+A*\text{Csc}[c+d*x]),x]$

[Out]  $-\left(\frac{aA \operatorname{ArcTanh}[\cos[c+d*x]]}{d}\right) - \frac{5*a*A*\cot[c+d*x]}{(3*d)} - \frac{a*A*\cot[c+d*x]*\csc[c+d*x]}{d} - \frac{a*A*\cot[c+d*x]*\csc[c+d*x]^2}{(3*d)}$

#### Rule 21

$\text{Int}[(u_.)*((a_.)+(b_.)*(v_.))^{(m_.)*((c_.)+(d_.)*(v_.))^{(n_.)}, x\_Symbol] \rightarrow$   
 $\text{Dist}[(b/d)^m, \text{Int}[u*(c+d*v)^{(m+n)}, x], x] /;$   $\text{FreeQ}[\{a, b, c, d, n\}, x]$   
 $\&\& \text{EqQ}[b*c - a*d, 0] \&\& \text{IntegerQ}[m] \&\& (!\text{IntegerQ}[n] \parallel \text{SimplerQ}[c+d*x,$   
 $a+b*x])$

#### Rule 3788

$\text{Int}[(\csc[(e_.)+(f_.)*(x_.)]*(d_.))^{(n_.)*(\csc[(e_.)+(f_.)*(x_.)]*(b_.)+(a_.))^2, x\_Symbol] \rightarrow$   
 $\text{Dist}[(2*a*b)/d, \text{Int}[(d*\csc[e+f*x])^{(n+1)}, x], x]$   
 $+ \text{Int}[(d*\csc[e+f*x])^n*(a^2+b^2*\csc[e+f*x]^2), x] /;$   $\text{FreeQ}[\{a, b, d, e, f, n\}, x]$

#### Rule 3768

$\text{Int}[(\csc[(c_.)+(d_.)*(x_.)]*(b_.))^{(n_.)}, x\_Symbol] \rightarrow$   
 $-\text{Simp}[(b*\cos[c+d*x])*(b*\csc[c+d*x])^{(n-1)}/(d*(n-1)), x] + \text{Dist}[(b^2*(n-2))/(n-1), I$

Int[(b\*Csc[c + d\*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2\*n]

### Rule 3770

Int[csc[(c\_.) + (d\_.)\*(x\_.)], x\_Symbol] := -Simp[ArcTanh[Cos[c + d\*x]]/d, x] /; FreeQ[{c, d}, x]

### Rule 4046

Int[(csc[(e\_.) + (f\_.)\*(x\_.)]\*(b\_.))^(m\_.)\*(csc[(e\_.) + (f\_.)\*(x\_.)]^2\*(C\_.) + (A\_.)), x\_Symbol] := -Simp[(C\*Cot[e + f\*x]\*(b\*Csc[e + f\*x])^m)/(f\*(m + 1)), x] + Dist[(C\*m + A\*(m + 1))/(m + 1), Int[(b\*Csc[e + f\*x])^m, x], x] /; FreeQ[{b, e, f, A, C, m}, x] && NeQ[C\*m + A\*(m + 1), 0] && !LeQ[m, -1]

### Rule 3767

Int[csc[(c\_.) + (d\_.)\*(x\_.)]^(n\_.), x\_Symbol] := -Dist[d^(-1), Subst[Int[ExpandIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d\*x]], x] /; FreeQ[{c, d}, x] && IGtQ[n/2, 0]

### Rule 8

Int[a\_, x\_Symbol] := Simp[a\*x, x] /; FreeQ[a, x]

### Rubi steps

$$\begin{aligned}
 \int \csc^2(c + dx)(a + a \csc(c + dx))(A + A \csc(c + dx)) dx &= \frac{A \int \csc^2(c + dx)(a + a \csc(c + dx))^2 dx}{a} \\
 &= \frac{A \int \csc^2(c + dx) (a^2 + a^2 \csc^2(c + dx)) dx}{a} + (2aA) \int \csc^3 \\
 &= -\frac{aA \cot(c + dx) \csc(c + dx)}{d} - \frac{aA \cot(c + dx) \csc^2(c + dx)}{3d} \\
 &= -\frac{aA \tanh^{-1}(\cos(c + dx))}{d} - \frac{aA \cot(c + dx) \csc(c + dx)}{d} - \frac{aA \cot(c + dx) \csc^2(c + dx)}{3d} \\
 &= -\frac{aA \tanh^{-1}(\cos(c + dx))}{d} - \frac{5aA \cot(c + dx)}{3d} - \frac{aA \cot(c + dx) \csc(c + dx)}{d}
 \end{aligned}$$

**Mathematica [A]** time = 0.0448655, size = 116, normalized size = 1.61

$$\frac{5aA \cot(c + dx)}{3d} - \frac{aA \csc^2\left(\frac{1}{2}(c + dx)\right)}{4d} + \frac{aA \sec^2\left(\frac{1}{2}(c + dx)\right)}{4d} + \frac{aA \log\left(\sin\left(\frac{1}{2}(c + dx)\right)\right)}{d} - \frac{aA \log\left(\cos\left(\frac{1}{2}(c + dx)\right)\right)}{d}$$

Antiderivative was successfully verified.

[In] Integrate[Csc[c + d\*x]^2\*(a + a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x]),x]

[Out]  $(-5*a*A*\cot[c + d*x])/(3*d) - (a*A*Csc[(c + d*x)/2]^2)/(4*d) - (a*A*\cot[c + d*x]*Csc[c + d*x]^2)/(3*d) - (a*A*\log[\cos[(c + d*x)/2]])/d + (a*A*\log[\sin[(c + d*x)/2]])/d + (a*A*\sec[(c + d*x)/2]^2)/(4*d)$

**Maple [A]** time = 0.043, size = 77, normalized size = 1.1

$$-\frac{5 A a \cot (d x+c)}{3 d}-\frac{A a \cot (d x+c) \csc (d x+c)}{d}+\frac{A a \ln (\csc (d x+c)-\cot (d x+c))}{d}-\frac{A a \cot (d x+c)(\csc (d x+c))^2}{3 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int(csc(d\*x+c)^2\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x)

[Out]  $-5/3*a*A*\cot(d*x+c)/d - a*A*\cot(d*x+c)*\csc(d*x+c)/d + 1/d*A*a*\ln(\csc(d*x+c) - \cot(d*x+c)) - 1/3*a*A*\cot(d*x+c)*\csc(d*x+c)^2/d$

**Maxima [A]** time = 0.987549, size = 117, normalized size = 1.62

$$\frac{3 A a \left( \frac{2 \cos (d x+c)}{\cos (d x+c)^2-1} - \log (\cos (d x+c)+1) + \log (\cos (d x+c)-1) \right) - \frac{6 A a}{\tan (d x+c)} - \frac{2(3 \tan (d x+c)^2+1) A a}{\tan (d x+c)^3}}{6 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^2\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="maxima")

[Out]  $1/6*(3*A*a*(2*\cos(d*x + c)/(\cos(d*x + c)^2 - 1) - \log(\cos(d*x + c) + 1) + \log(\cos(d*x + c) - 1)) - 6*A*a/\tan(d*x + c) - 2*(3*\tan(d*x + c)^2 + 1)*A*a/\tan(d*x + c)^3)/d$

**Fricas [A]** time = 0.500348, size = 363, normalized size = 5.04

$$\frac{10 A a \cos (d x+c)^3 - 6 A a \cos (d x+c) \sin (d x+c) - 12 A a \cos (d x+c) + 3 \left( A a \cos (d x+c)^2 - A a \right) \log \left( \frac{1}{2} \cos (d x+c) \right)}{6 \left( d \cos (d x+c)^2 - d \right) \sin (d x+c)}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^2\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="fricas")

[Out] 
$$-1/6*(10*A*a*\cos(d*x + c)^3 - 6*A*a*\cos(d*x + c)*\sin(d*x + c) - 12*A*a*\cos(d*x + c) + 3*(A*a*\cos(d*x + c)^2 - A*a)*\log(1/2*\cos(d*x + c) + 1/2)*\sin(d*x + c) - 3*(A*a*\cos(d*x + c)^2 - A*a)*\log(-1/2*\cos(d*x + c) + 1/2)*\sin(d*x + c))/((d*\cos(d*x + c)^2 - d)*\sin(d*x + c))$$

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$Aa \left( \int \csc^2(c + dx) dx + \int 2 \csc^3(c + dx) dx + \int \csc^4(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*\*2\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x)

[Out]  $A*a*(\text{Integral}(\csc(c + d*x)**2, x) + \text{Integral}(2*\csc(c + d*x)**3, x) + \text{Integral}(\csc(c + d*x)**4, x))$

**Giac [A]** time = 1.35668, size = 166, normalized size = 2.31

$$Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^3 + 6 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^2 + 24 Aa \log\left(\left|\tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)\right|\right) + 21 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right) - \frac{44 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)}{24 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^2\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="giac")

[Out] 
$$1/24*(A*a*\tan(1/2*d*x + 1/2*c)^3 + 6*A*a*\tan(1/2*d*x + 1/2*c)^2 + 24*A*a*\log(\text{abs}(\tan(1/2*d*x + 1/2*c))) + 21*A*a*\tan(1/2*d*x + 1/2*c) - (44*A*a*\tan(1/2*d*x + 1/2*c)^3 + 21*A*a*\tan(1/2*d*x + 1/2*c)^2 + 6*A*a*\tan(1/2*d*x + 1/2*c) + A*a)/\tan(1/2*d*x + 1/2*c)^3)/d$$

### 3.3 $\int \csc(c+dx)(a+a \csc(c+dx))(A+A \csc(c+dx)) dx$

**Optimal.** Leaf size=51

$$-\frac{2aA \cot(c+dx)}{d} - \frac{3aA \tanh^{-1}(\cos(c+dx))}{2d} - \frac{aA \cot(c+dx) \csc(c+dx)}{2d}$$

[Out]  $(-3*a*A*ArcTanh[Cos[c + d*x]])/(2*d) - (2*a*A*Cot[c + d*x])/d - (a*A*Cot[c + d*x]*Csc[c + d*x])/(2*d)$

**Rubi [A]** time = 0.0576248, antiderivative size = 51, normalized size of antiderivative = 1., number of steps used = 6, number of rules used = 6, integrand size = 27,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.222$ , Rules used = {21, 3788, 3767, 8, 4046, 3770}

$$-\frac{2aA \cot(c+dx)}{d} - \frac{3aA \tanh^{-1}(\cos(c+dx))}{2d} - \frac{aA \cot(c+dx) \csc(c+dx)}{2d}$$

Antiderivative was successfully verified.

[In]  $\text{Int}[Csc[c + d*x]*(a + a*Csc[c + d*x])*(A + A*Csc[c + d*x]),x]$

[Out]  $(-3*a*A*ArcTanh[Cos[c + d*x]])/(2*d) - (2*a*A*Cot[c + d*x])/d - (a*A*Cot[c + d*x]*Csc[c + d*x])/(2*d)$

#### Rule 21

$\text{Int}[(u_.)*((a_.) + (b_.)*(v_.))^(m_.)*((c_.) + (d_.)*(v_.))^(n_.), x\_Symbol] \rightarrow \text{Dist}[(b/d)^m, \text{Int}[u*(c + d*v)^(m + n), x], x] /; \text{FreeQ}[\{a, b, c, d, n\}, x] \&\& \text{EqQ}[b*c - a*d, 0] \&\& \text{IntegerQ}[m] \&\& (!\text{IntegerQ}[n] || \text{SimplerQ}[c + d*x, a + b*x])$

#### Rule 3788

$\text{Int}[(\csc[(e_.) + (f_.)*(x_.)]*(d_.))^(n_.)*(\csc[(e_.) + (f_.)*(x_.)]*(b_.) + (a_.))^2, x\_Symbol] \rightarrow \text{Dist}[(2*a*b)/d, \text{Int}[(d*Csc[e + f*x])^(n + 1), x], x] + \text{Int}[(d*Csc[e + f*x])^n*(a^2 + b^2*Csc[e + f*x]^2), x] /; \text{FreeQ}[\{a, b, d, e, f, n\}, x]$

#### Rule 3767

$\text{Int}[\csc[(c_.) + (d_.)*(x_.)]^(n_.), x\_Symbol] \rightarrow -\text{Dist}[d^(-1), \text{Subst}[\text{Int}[\text{ExpandIntegrand}[(1 + x^2)^(n/2 - 1), x], x], x, \text{Cot}[c + d*x]], x] /; \text{FreeQ}[\{c,$



d}, x] && IGtQ[n/2, 0]

### Rule 8

Int[a\_, x\_Symbol] := Simp[a\*x, x] /; FreeQ[a, x]

### Rule 4046

Int[(csc[(e\_.) + (f\_.)\*(x\_.)]\*(b\_.))^(m\_.)\*(csc[(e\_.) + (f\_.)\*(x\_.)]^2\*(C\_.) + (A\_.)), x\_Symbol] := -Simp[(C\*Cot[e + f\*x]\*(b\*Csc[e + f\*x])^m)/(f\*(m + 1)), x] + Dist[(C\*m + A\*(m + 1))/(m + 1), Int[(b\*Csc[e + f\*x])^m, x], x] /; FreeQ[{b, e, f, A, C, m}, x] && NeQ[C\*m + A\*(m + 1), 0] && !LeQ[m, -1]

### Rule 3770

Int[csc[(c\_.) + (d\_.)\*(x\_.)], x\_Symbol] := -Simp[ArcTanh[Cos[c + d\*x]]/d, x] /; FreeQ[{c, d}, x]

### Rubi steps

$$\begin{aligned} \int \csc(c + dx)(a + a \csc(c + dx))(A + A \csc(c + dx)) dx &= \frac{A \int \csc(c + dx)(a + a \csc(c + dx))^2 dx}{a} \\ &= \frac{A \int \csc(c + dx)(a^2 + a^2 \csc^2(c + dx)) dx}{a} + (2aA) \int \csc^2(c + dx) dx \\ &= -\frac{aA \cot(c + dx) \csc(c + dx)}{2d} + \frac{1}{2}(3aA) \int \csc(c + dx) dx - \frac{aA \csc^2(c + dx)}{2d} \\ &= -\frac{3aA \tanh^{-1}(\cos(c + dx))}{2d} - \frac{2aA \cot(c + dx)}{d} - \frac{aA \cot(c + dx)}{d} \end{aligned}$$

**Mathematica [B]** time = 0.0399808, size = 137, normalized size = 2.69

$$-\frac{2aA \cot(c + dx)}{d} - \frac{aA \csc^2\left(\frac{1}{2}(c + dx)\right)}{8d} + \frac{aA \sec^2\left(\frac{1}{2}(c + dx)\right)}{8d} + \frac{aA \log\left(\sin\left(\frac{c}{2} + \frac{dx}{2}\right)\right)}{d} + \frac{aA \log\left(\sin\left(\frac{1}{2}(c + dx)\right)\right)}{2d}$$

Antiderivative was successfully verified.

[In] Integrate[Csc[c + d\*x]\*(a + a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x]), x]

[Out] (-2\*a\*A\*Cot[c + d\*x])/d - (a\*A\*Csc[(c + d\*x)/2]^2)/(8\*d) - (a\*A\*Log[Cos[c/2 + (d\*x)/2]])/d - (a\*A\*Log[Cos[(c + d\*x)/2]])/(2\*d) + (a\*A\*Log[Sin[c/2 + (d\*x)/2]])/d

$\ast x)/2]])/d + (a\ast A\ast \text{Log}[\text{Sin}[(c + d\ast x)/2]])/(2\ast d) + (a\ast A\ast \text{Sec}[(c + d\ast x)/2]^2)/(8\ast d)$

**Maple [A]** time = 0.045, size = 57, normalized size = 1.1

$$\frac{3 A a \ln (\csc (d x+c)-\cot (d x+c))}{2 d}-2 \frac{A a \cot (d x+c)}{d}-\frac{A a \cot (d x+c) \csc (d x+c)}{2 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int(csc(d\*x+c)\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x)

[Out] 3/2/d\*A\*a\*ln(csc(d\*x+c)-cot(d\*x+c))-2\*a\*A\*cot(d\*x+c)/d-1/2\*a\*A\*cot(d\*x+c)\*csc(d\*x+c)/d

**Maxima [A]** time = 0.999313, size = 108, normalized size = 2.12

$$\frac{A a \left( \frac{2 \cos (d x+c)}{\cos (d x+c)^2-1} - \log (\cos (d x+c)+1) + \log (\cos (d x+c)-1) \right) - 4 A a \log (\cot (d x+c)+\csc (d x+c)) - \frac{8 A a}{\tan (d x+c)}}{4 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="maxima")

[Out] 1/4\*(A\*a\*(2\*cos(d\*x + c)/(cos(d\*x + c)^2 - 1) - log(cos(d\*x + c) + 1) + log(cos(d\*x + c) - 1)) - 4\*A\*a\*log(cot(d\*x + c) + csc(d\*x + c)) - 8\*A\*a/tan(d\*x + c))/d

**Fricas [B]** time = 0.49257, size = 273, normalized size = 5.35

$$\frac{8 A a \cos (d x+c) \sin (d x+c)+2 A a \cos (d x+c)-3\left(A a \cos (d x+c)^2-A a\right) \log \left(\frac{1}{2} \cos (d x+c)+\frac{1}{2}\right)+3\left(A a \cos (d x+c)\right)}{4\left(d \cos (d x+c)^2-d\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="fricas")

[Out]  $\frac{1}{4}*(8*A*a*\cos(d*x + c)*\sin(d*x + c) + 2*A*a*\cos(d*x + c) - 3*(A*a*\cos(d*x + c)^2 - A*a)*\log(1/2*\cos(d*x + c) + 1/2) + 3*(A*a*\cos(d*x + c)^2 - A*a)*\log(-1/2*\cos(d*x + c) + 1/2))/(d*\cos(d*x + c)^2 - d)$

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$Aa \left( \int \csc(c + dx) dx + \int 2 \csc^2(c + dx) dx + \int \csc^3(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x)

[Out]  $A*a*(\text{Integral}(\csc(c + d*x), x) + \text{Integral}(2*\csc(c + d*x)**2, x) + \text{Integral}(\csc(c + d*x)**3, x))$

**Giac [A]** time = 1.36763, size = 126, normalized size = 2.47

$$\frac{Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^2 + 12 Aa \log\left(\left|\tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)\right|\right) + 8 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right) - \frac{18 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^2 + 8 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right) + Aa}{\tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^2}}{8d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="giac")

[Out]  $\frac{1}{8}*(A*a*\tan(1/2*d*x + 1/2*c)^2 + 12*A*a*\log(\text{abs}(\tan(1/2*d*x + 1/2*c))) + 8*A*a*\tan(1/2*d*x + 1/2*c) - (18*A*a*\tan(1/2*d*x + 1/2*c)^2 + 8*A*a*\tan(1/2*d*x + 1/2*c) + A*a)/\tan(1/2*d*x + 1/2*c)^2)/d$

### 3.4 $\int (a + a \csc(c + dx))(A + A \csc(c + dx)) \sin(c + dx) dx$

Optimal. Leaf size=33

$$-\frac{aA \cos(c + dx)}{d} - \frac{aA \tanh^{-1}(\cos(c + dx))}{d} + 2aAx$$

[Out]  $2*a*A*x - (a*A*ArcTanh[Cos[c + d*x]])/d - (a*A*Cos[c + d*x])/d$

**Rubi [A]** time = 0.0594924, antiderivative size = 33, normalized size of antiderivative = 1., number of steps used = 5, number of rules used = 5, integrand size = 27,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.185$ , Rules used = {21, 3788, 8, 4045, 3770}

$$-\frac{aA \cos(c + dx)}{d} - \frac{aA \tanh^{-1}(\cos(c + dx))}{d} + 2aAx$$

Antiderivative was successfully verified.

[In]  $\text{Int}[(a + a*\text{Csc}[c + d*x])*(A + A*\text{Csc}[c + d*x])*Sin[c + d*x], x]$

[Out]  $2*a*A*x - (a*A*ArcTanh[Cos[c + d*x]])/d - (a*A*Cos[c + d*x])/d$

#### Rule 21

$\text{Int}[(u_.)*((a_.) + (b_.)*(v_))^{(m_.)}*((c_.) + (d_.)*(v_))^{(n_.)}, x\_Symbol] :>$   
 $\text{Dist}[(b/d)^m, \text{Int}[u*(c + d*v)^{(m + n)}, x], x] /;$   $\text{FreeQ}[\{a, b, c, d, n\}, x]$   
 $\&\& \text{EqQ}[b*c - a*d, 0] \&\& \text{IntegerQ}[m] \&\& (!\text{IntegerQ}[n] || \text{SimplerQ}[c + d*x,$   
 $a + b*x])$

#### Rule 3788

$\text{Int}[(\text{csc}[(e_.) + (f_.)*(x_)]*(d_.)^{(n_.)}*(\text{csc}[(e_.) + (f_.)*(x_)]*(b_.) +$   
 $(a_.)^2, x\_Symbol] :> \text{Dist}[(2*a*b)/d, \text{Int}[(d*\text{Csc}[e + f*x])^{(n + 1)}, x], x]$   
 $+ \text{Int}[(d*\text{Csc}[e + f*x])^n*(a^2 + b^2*\text{Csc}[e + f*x]^2), x] /;$   $\text{FreeQ}[\{a, b, d,$   
 $e, f, n\}, x]$

#### Rule 8

$\text{Int}[a_, x\_Symbol] :> \text{Simp}[a*x, x] /;$   $\text{FreeQ}[a, x]$

#### Rule 4045

```
Int[(csc[(e_.) + (f_.)*(x_.)]*(b_.))^(m_.)*(csc[(e_.) + (f_.)*(x_.)]^2*(C_.)
+ (A_.)), x_Symbol] := Simp[(A*Cot[e + f*x]*(b*Csc[e + f*x])^m)/(f*m), x] +
Dist[(C*m + A*(m + 1))/(b^2*m), Int[(b*Csc[e + f*x])^(m + 2), x], x] /; FreeQ[{b, e, f, A, C}, x] && NeQ[C*m + A*(m + 1), 0] && LeQ[m, -1]
```

### Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_.)], x_Symbol] := -Simp[ArcTanh[Cos[c + d*x]]/d, x]
/; FreeQ[{c, d}, x]
```

### Rubi steps

$$\begin{aligned} \int (a + a \csc(c + dx))(A + A \csc(c + dx)) \sin(c + dx) dx &= \frac{A \int (a + a \csc(c + dx))^2 \sin(c + dx) dx}{a} \\ &= \frac{A \int (a^2 + a^2 \csc^2(c + dx)) \sin(c + dx) dx}{a} + (2aA) \int 1 dx \\ &= 2aAx - \frac{aA \cos(c + dx)}{d} + (aA) \int \csc(c + dx) dx \\ &= 2aAx - \frac{aA \tanh^{-1}(\cos(c + dx))}{d} - \frac{aA \cos(c + dx)}{d} \end{aligned}$$

**Mathematica [B]** time = 0.0225814, size = 72, normalized size = 2.18

$$\frac{aA \sin(c) \sin(dx)}{d} - \frac{aA \cos(c) \cos(dx)}{d} + \frac{aA \log\left(\sin\left(\frac{c}{2} + \frac{dx}{2}\right)\right)}{d} - \frac{aA \log\left(\cos\left(\frac{c}{2} + \frac{dx}{2}\right)\right)}{d} + 2aAx$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + a*Csc[c + d*x])*(A + A*Csc[c + d*x])*Sin[c + d*x], x]
```

```
[Out] 2*a*A*x - (a*A*Cos[c]*Cos[d*x])/d - (a*A*Log[Cos[c/2 + (d*x)/2]])/d + (a*A*
Log[Sin[c/2 + (d*x)/2]])/d + (a*A*Sin[c]*Sin[d*x])/d
```

**Maple [A]** time = 0.079, size = 50, normalized size = 1.5

$$2aAx - \frac{Aa \cos(dx + c)}{d} + \frac{Aa \ln(\csc(dx + c) - \cot(dx + c))}{d} + 2 \frac{Aac}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `int((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c),x)`

[Out]  $2*a*A*x - a*A*\cos(d*x+c)/d + 1/d*A*a*\ln(\csc(d*x+c) - \cot(d*x+c)) + 2/d*A*a*c$

**Maxima [A]** time = 0.988842, size = 68, normalized size = 2.06

$$\frac{4(dx+c)Aa - Aa(\log(\cos(dx+c)+1) - \log(\cos(dx+c)-1)) - 2Aa\cos(dx+c)}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c),x, algorithm="maxima")`

[Out]  $1/2*(4*(d*x+c)*A*a - A*a*(\log(\cos(d*x+c)+1) - \log(\cos(d*x+c)-1)) - 2*A*a*\cos(d*x+c))/d$

**Fricas [A]** time = 0.507614, size = 147, normalized size = 4.45

$$\frac{4Aadx - 2Aa\cos(dx+c) - Aa\log\left(\frac{1}{2}\cos(dx+c) + \frac{1}{2}\right) + Aa\log\left(-\frac{1}{2}\cos(dx+c) + \frac{1}{2}\right)}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c),x, algorithm="fricas")`

[Out]  $1/2*(4*A*a*d*x - 2*A*a*\cos(d*x+c) - A*a*\log(1/2*\cos(d*x+c) + 1/2) + A*a*\log(-1/2*\cos(d*x+c) + 1/2))/d$

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$Aa\left(\int 2\sin(c+dx)\csc(c+dx)dx + \int \sin(c+dx)\csc^2(c+dx)dx + \int \sin(c+dx)dx\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c),x)
```

```
[Out] A*a*(Integral(2*sin(c + d*x)*csc(c + d*x), x) + Integral(sin(c + d*x)*csc(c
+ d*x)**2, x) + Integral(sin(c + d*x), x))
```

**Giac [A]** time = 1.35554, size = 63, normalized size = 1.91

$$\frac{2(dx+c)Aa + Aa \log\left(\left|\tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)\right|\right) - \frac{2Aa}{\tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^2 + 1}}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c),x, algorithm="giac")
```

```
[Out] (2*(d*x + c)*A*a + A*a*log(abs(tan(1/2*d*x + 1/2*c)))) - 2*A*a/(tan(1/2*d*x
+ 1/2*c)^2 + 1))/d
```

### 3.5 $\int (a + a \csc(c + dx))(A + A \csc(c + dx)) \sin^2(c + dx) dx$

**Optimal.** Leaf size=42

$$-\frac{2aA \cos(c + dx)}{d} - \frac{aA \sin(c + dx) \cos(c + dx)}{2d} + \frac{3aAx}{2}$$

[Out] (3\*a\*A\*x)/2 - (2\*a\*A\*Cos[c + d\*x])/d - (a\*A\*Cos[c + d\*x]\*Sin[c + d\*x])/(2\*d)

**Rubi [A]** time = 0.0733261, antiderivative size = 42, normalized size of antiderivative = 1., number of steps used = 5, number of rules used = 5, integrand size = 29,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.172$ , Rules used = {21, 3788, 2638, 4045, 8}

$$-\frac{2aA \cos(c + dx)}{d} - \frac{aA \sin(c + dx) \cos(c + dx)}{2d} + \frac{3aAx}{2}$$

Antiderivative was successfully verified.

[In] Int[(a + a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x])\*Sin[c + d\*x]^2,x]

[Out] (3\*a\*A\*x)/2 - (2\*a\*A\*Cos[c + d\*x])/d - (a\*A\*Cos[c + d\*x]\*Sin[c + d\*x])/(2\*d)

#### Rule 21

```
Int[(u_.)*((a_) + (b_.)*(v_))^(m_.)*((c_) + (d_.)*(v_))^(n_.), x_Symbol] :>
  Dist[(b/d)^m, Int[u*(c + d*v)^(m + n), x], x] /; FreeQ[{a, b, c, d, n}, x]
  && EqQ[b*c - a*d, 0] && IntegerQ[m] && (!IntegerQ[n] || SimplerQ[c + d*x,
  a + b*x])
```

#### Rule 3788

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_.))^2, x_Symbol] :> Dist[(2*a*b)/d, Int[(d*Csc[e + f*x])^(n + 1), x], x]
+ Int[(d*Csc[e + f*x])^n*(a^2 + b^2*Csc[e + f*x]^2), x] /; FreeQ[{a, b, d,
e, f, n}, x]
```

#### Rule 2638

```
Int[sin[(c_.) + (d_.)*(x_)], x_Symbol] :> -Simp[Cos[c + d*x]/d, x] /; FreeQ
[{c, d}, x]
```



Rule 4045

Int[(csc[(e\_.) + (f\_.)\*(x\_.)]\*(b\_.))^(m\_.)\*(csc[(e\_.) + (f\_.)\*(x\_.)]^2\*(C\_.) + (A\_.)), x\_Symbol] :> Simp[(A\*Cot[e + f\*x]\*(b\*Csc[e + f\*x])^m)/(f\*m), x] + Dist[(C\*m + A\*(m + 1))/(b^2\*m), Int[(b\*Csc[e + f\*x])^(m + 2), x], x] /; FreeQ[{b, e, f, A, C}, x] && NeQ[C\*m + A\*(m + 1), 0] && LeQ[m, -1]

Rule 8

Int[a\_, x\_Symbol] :> Simp[a\*x, x] /; FreeQ[a, x]

Rubi steps

$$\begin{aligned} \int (a + a \csc(c + dx))(A + A \csc(c + dx)) \sin^2(c + dx) dx &= \frac{A \int (a + a \csc(c + dx))^2 \sin^2(c + dx) dx}{a} \\ &= \frac{A \int (a^2 + a^2 \csc^2(c + dx)) \sin^2(c + dx) dx}{a} + (2aA) \int \sin(c + dx) dx \\ &= -\frac{2aA \cos(c + dx)}{d} - \frac{aA \cos(c + dx) \sin(c + dx)}{2d} + \frac{1}{2}(3aA) \int \sin(c + dx) dx \\ &= \frac{3aAx}{2} - \frac{2aA \cos(c + dx)}{d} - \frac{aA \cos(c + dx) \sin(c + dx)}{2d} \end{aligned}$$

**Mathematica [A]** time = 0.0632426, size = 33, normalized size = 0.79

$$-\frac{aA(-6(c + dx) + \sin(2(c + dx)) + 8 \cos(c + dx))}{4d}$$

Antiderivative was successfully verified.

[In] Integrate[(a + a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x])\*Sin[c + d\*x]^2,x]

[Out] -(a\*A\*(-6\*(c + d\*x) + 8\*Cos[c + d\*x] + Sin[2\*(c + d\*x)]))/(4\*d)

**Maple [A]** time = 0.078, size = 49, normalized size = 1.2

$$\frac{1}{d} \left( Aa \left( -\frac{\cos(dx + c) \sin(dx + c)}{2} + \frac{dx}{2} + \frac{c}{2} \right) - 2Aa \cos(dx + c) + Aa(dx + c) \right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c)^2,x)
```

```
[Out] 1/d*(A*a*(-1/2*cos(d*x+c)*sin(d*x+c)+1/2*d*x+1/2*c)-2*A*a*cos(d*x+c)+A*a*(d*x+c))
```

**Maxima [A]** time = 0.990114, size = 63, normalized size = 1.5

$$\frac{(2 dx + 2 c - \sin(2 dx + 2 c))Aa + 4 (dx + c)Aa - 8 Aa \cos(dx + c)}{4 d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c)^2,x, algorithm="maxima")
```

```
[Out] 1/4*((2*d*x + 2*c - sin(2*d*x + 2*c))*A*a + 4*(d*x + c)*A*a - 8*A*a*cos(d*x + c))/d
```

**Fricas [A]** time = 0.476211, size = 97, normalized size = 2.31

$$\frac{3 Aa dx - Aa \cos(dx + c) \sin(dx + c) - 4 Aa \cos(dx + c)}{2 d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c)^2,x, algorithm="fricas")
```

```
[Out] 1/2*(3*A*a*d*x - A*a*cos(d*x + c)*sin(d*x + c) - 4*A*a*cos(d*x + c))/d
```

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$Aa \left( \int 2 \sin^2(c + dx) \csc(c + dx) dx + \int \sin^2(c + dx) \csc^2(c + dx) dx + \int \sin^2(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c)**2,x)
```

[Out]  $A*a*(\text{Integral}(2*\sin(c + d*x)**2*\csc(c + d*x), x) + \text{Integral}(\sin(c + d*x)**2*\csc(c + d*x)**2, x) + \text{Integral}(\sin(c + d*x)**2, x))$

**Giac [B]** time = 1.49724, size = 107, normalized size = 2.55

$$\frac{3(dx + c)Aa + \frac{2\left(Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^3 - 4Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^2 - Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right) - 4Aa\right)}{\left(\tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^2 + 1\right)^2}}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c)^2,x, algorithm="giac")`

[Out]  $\frac{1}{2}*(3*(d*x + c)*A*a + 2*(A*a*\tan(1/2*d*x + 1/2*c)^3 - 4*A*a*\tan(1/2*d*x + 1/2*c)^2 - A*a*\tan(1/2*d*x + 1/2*c) - 4*A*a)/(\tan(1/2*d*x + 1/2*c)^2 + 1)^2)/d$

### 3.6 $\int (a + a \csc(c + dx))(A + A \csc(c + dx)) \sin^3(c + dx) dx$

**Optimal.** Leaf size=54

$$\frac{aA \cos^3(c + dx)}{3d} - \frac{2aA \cos(c + dx)}{d} - \frac{aA \sin(c + dx) \cos(c + dx)}{d} + aAx$$

[Out]  $aAx - (2aA \cos(c + dx))/d + (aA \cos^3(c + dx))/(3d) - (aA \cos(c + dx) \sin(c + dx))/d$

**Rubi [A]** time = 0.0940265, antiderivative size = 54, normalized size of antiderivative = 1., number of steps used = 7, number of rules used = 6, integrand size = 29,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.207$ , Rules used = {21, 3788, 2635, 8, 4044, 3013}

$$\frac{aA \cos^3(c + dx)}{3d} - \frac{2aA \cos(c + dx)}{d} - \frac{aA \sin(c + dx) \cos(c + dx)}{d} + aAx$$

Antiderivative was successfully verified.

[In]  $\text{Int}[(a + a \csc(c + dx))(A + A \csc(c + dx)) \sin^3(c + dx), x]$

[Out]  $aAx - (2aA \cos(c + dx))/d + (aA \cos^3(c + dx))/(3d) - (aA \cos(c + dx) \sin(c + dx))/d$

#### Rule 21

```
Int[(u_)*((a_) + (b_)*(v_))^(m_)*((c_) + (d_)*(v_))^(n_), x_Symbol] :=
  Dist[(b/d)^m, Int[u*(c + d*v)^(m + n), x], x] /; FreeQ[{a, b, c, d, n}, x]
  && EqQ[b*c - a*d, 0] && IntegerQ[m] && (!IntegerQ[n] || SimplerQ[c + d*x,
  a + b*x])
```

#### Rule 3788

```
Int[(csc[(e_) + (f_)*(x_)]*(d_))^(n_)*(csc[(e_) + (f_)*(x_)]*(b_) +
(a_))^(2), x_Symbol] := Dist[(2*a*b)/d, Int[(d*Csc[e + f*x])^(n + 1), x], x]
+ Int[(d*Csc[e + f*x])^n*(a^2 + b^2*Csc[e + f*x]^2), x] /; FreeQ[{a, b, d,
e, f, n}, x]
```

#### Rule 2635

```
Int[((b_)*sin[(c_) + (d_)*(x_)])^(n_), x_Symbol] := -Simp[(b*Cos[c + d*x]
)*(b*Sin[c + d*x])^(n - 1))/(d*n), x] + Dist[(b^2*(n - 1))/n, Int[(b*Sin[c
```

+ d\*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2\*n]

### Rule 8

Int[a\_, x\_Symbol] := Simp[a\*x, x] /; FreeQ[a, x]

### Rule 4044

Int[csc[(e\_.) + (f\_.)\*(x\_)]^(m\_.)\*(csc[(e\_.) + (f\_.)\*(x\_)]^2\*(C\_.) + (A\_.)),  
x\_Symbol] := Int[(C + A\*Sin[e + f\*x]^2)/Sin[e + f\*x]^(m + 2), x] /; FreeQ[  
{e, f, A, C}, x] && NeQ[C\*m + A\*(m + 1), 0] && ILtQ[(m + 1)/2, 0]

### Rule 3013

Int[sin[(e\_.) + (f\_.)\*(x\_)]^(m\_.)\*((A\_) + (C\_.)\*sin[(e\_.) + (f\_.)\*(x\_)]^2),  
x\_Symbol] := -Dist[f^(-1), Subst[Int[(1 - x^2)^((m - 1)/2)\*(A + C - C\*x^2)]  
, x], x, Cos[e + f\*x]], x] /; FreeQ[{e, f, A, C}, x] && IGtQ[(m + 1)/2, 0]

### Rubi steps

$$\begin{aligned}
 \int (a + a \csc(c + dx))(A + A \csc(c + dx)) \sin^3(c + dx) dx &= \frac{A \int (a + a \csc(c + dx))^2 \sin^3(c + dx) dx}{a} \\
 &= \frac{A \int (a^2 + a^2 \csc^2(c + dx)) \sin^3(c + dx) dx}{a} + (2aA) \int \sin^2(c + dx) dx \\
 &= -\frac{aA \cos(c + dx) \sin(c + dx)}{d} + \frac{A \int \sin(c + dx) (a^2 + a^2 \sin^2(c + dx)) dx}{a} \\
 &= aAx - \frac{aA \cos(c + dx) \sin(c + dx)}{d} - \frac{A \text{Subst}\left(\int (2a^2 - a^2x^2) dx\right)}{ad} \\
 &= aAx - \frac{2aA \cos(c + dx)}{d} + \frac{aA \cos^3(c + dx)}{3d} - \frac{aA \cos(c + dx)}{d}
 \end{aligned}$$

**Mathematica [A]** time = 0.0841558, size = 43, normalized size = 0.8

$$\frac{aA(-6 \sin(2(c + dx)) - 21 \cos(c + dx) + \cos(3(c + dx)) + 12c + 12dx)}{12d}$$

Antiderivative was successfully verified.

[In] Integrate[(a + a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x])\*Sin[c + d\*x]^3,x]

[Out] (a\*A\*(12\*c + 12\*d\*x - 21\*Cos[c + d\*x] + Cos[3\*(c + d\*x)] - 6\*Sin[2\*(c + d\*x) ])))/(12\*d)

**Maple [A]** time = 0.089, size = 62, normalized size = 1.2

$$\frac{1}{d} \left( -\frac{Aa(2 + (\sin(dx + c))^2) \cos(dx + c)}{3} + 2Aa(-1/2 \cos(dx + c) \sin(dx + c) + 1/2 dx + c/2) - Aa \cos(dx + c) \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int((a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c))\*sin(d\*x+c)^3,x)

[Out] 1/d\*(-1/3\*A\*a\*(2+sin(d\*x+c)^2)\*cos(d\*x+c)+2\*A\*a\*(-1/2\*cos(d\*x+c)\*sin(d\*x+c)+1/2\*d\*x+1/2\*c)-A\*a\*cos(d\*x+c))

**Maxima [A]** time = 0.96236, size = 81, normalized size = 1.5

$$\frac{2(\cos(dx + c)^3 - 3 \cos(dx + c))Aa + 3(2dx + 2c - \sin(2dx + 2c))Aa - 6Aa \cos(dx + c)}{6d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c))\*sin(d\*x+c)^3,x, algorithm="maxima")

[Out] 1/6\*(2\*(cos(d\*x + c)^3 - 3\*cos(d\*x + c))\*A\*a + 3\*(2\*d\*x + 2\*c - sin(2\*d\*x + 2\*c))\*A\*a - 6\*A\*a\*cos(d\*x + c))/d

**Fricas [A]** time = 0.504407, size = 128, normalized size = 2.37

$$\frac{Aa \cos(dx + c)^3 + 3Aadx - 3Aa \cos(dx + c) \sin(dx + c) - 6Aa \cos(dx + c)}{3d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a+a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c))\*sin(d\*x+c)^3,x, algorithm="fricas")

[Out]  $\frac{1}{3}*(A*a*\cos(d*x + c)^3 + 3*A*a*d*x - 3*A*a*\cos(d*x + c)*\sin(d*x + c) - 6*A*a*\cos(d*x + c))/d$

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$Aa \left( \int 2 \sin^3(c + dx) \csc(c + dx) dx + \int \sin^3(c + dx) \csc^2(c + dx) dx + \int \sin^3(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c)**3,x)`

[Out]  $A*a*(\text{Integral}(2*\sin(c + d*x)**3*\csc(c + d*x), x) + \text{Integral}(\sin(c + d*x)**3*\csc(c + d*x)**2, x) + \text{Integral}(\sin(c + d*x)**3, x))$

**Giac [A]** time = 1.39078, size = 128, normalized size = 2.37

$$\frac{3(dx + c)Aa + \frac{2 \left( 3Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^5 - 3Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^4 - 12Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^2 - 3Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right) - 5Aa \right)}{\left(\tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^2 + 1\right)^3}{3d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a+a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c)^3,x, algorithm="giac")`

[Out]  $\frac{1}{3}*(3*(d*x + c)*A*a + 2*(3*A*a*\tan(1/2*d*x + 1/2*c)^5 - 3*A*a*\tan(1/2*d*x + 1/2*c)^4 - 12*A*a*\tan(1/2*d*x + 1/2*c)^2 - 3*A*a*\tan(1/2*d*x + 1/2*c) - 5*A*a)/(\tan(1/2*d*x + 1/2*c)^2 + 1)^3)/d$

### 3.7 $\int \csc^3(c+dx)(a-a \csc(c+dx))(A+A \csc(c+dx)) dx$

**Optimal.** Leaf size=61

$$-\frac{aA \tanh^{-1}(\cos(c+dx))}{8d} + \frac{aA \cot(c+dx) \csc^3(c+dx)}{4d} - \frac{aA \cot(c+dx) \csc(c+dx)}{8d}$$

[Out]  $-(a*A*ArcTanh[Cos[c + d*x]])/(8*d) - (a*A*Cot[c + d*x]*Csc[c + d*x])/(8*d) + (a*A*Cot[c + d*x]*Csc[c + d*x]^3)/(4*d)$

**Rubi [A]** time = 0.0924962, antiderivative size = 61, normalized size of antiderivative = 1., number of steps used = 4, number of rules used = 4, integrand size = 30,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.133$ , Rules used = {3962, 2611, 3768, 3770}

$$-\frac{aA \tanh^{-1}(\cos(c+dx))}{8d} + \frac{aA \cot(c+dx) \csc^3(c+dx)}{4d} - \frac{aA \cot(c+dx) \csc(c+dx)}{8d}$$

Antiderivative was successfully verified.

[In]  $\text{Int}[Csc[c + d*x]^3*(a - a*Csc[c + d*x])*(A + A*Csc[c + d*x]),x]$

[Out]  $-(a*A*ArcTanh[Cos[c + d*x]])/(8*d) - (a*A*Cot[c + d*x]*Csc[c + d*x])/(8*d) + (a*A*Cot[c + d*x]*Csc[c + d*x]^3)/(4*d)$

#### Rule 3962

$\text{Int}[(\csc[(e_.) + (f_.)*(x_.)]*(g_.))^{(p_.)}*(\csc[(e_.) + (f_.)*(x_.)]*(b_.) + (a_.))^{(m_.)}*(\csc[(e_.) + (f_.)*(x_.)]*(d_.) + (c_.))^{(n_.)}, x\_Symbol] \rightarrow \text{Dist}[(-a*c)^m, \text{Int}[\text{ExpandTrig}[(g*\csc[e + f*x])^p*\cot[e + f*x]^{(2*m)}, (c + d*\csc[e + f*x])^{(n - m)}, x], x], x] /;$  FreeQ[{a, b, c, d, e, f, g, n, p}, x] & & EqQ[b\*c + a\*d, 0] & & EqQ[a^2 - b^2, 0] & & IntegersQ[m, n] & & GeQ[n - m, 0] & & GtQ[m\*n, 0]

#### Rule 2611

$\text{Int}[(a_.)*\sec[(e_.) + (f_.)*(x_.)]^{(m_.)}*((b_.)*\tan[(e_.) + (f_.)*(x_.)]^{(n_.)}, x\_Symbol] \rightarrow \text{Simp}[(b*(a*\sec[e + f*x])^m*(b*\tan[e + f*x])^{(n - 1)})/(f*(m + n - 1)), x] - \text{Dist}[(b^2*(n - 1))/(m + n - 1), \text{Int}[(a*\sec[e + f*x])^m*(b*\tan[e + f*x])^{(n - 2)}, x], x] /;$  FreeQ[{a, b, e, f, m}, x] & & GtQ[n, 1] & & NeQ[m + n - 1, 0] & & IntegersQ[2\*m, 2\*n]

#### Rule 3768



```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := -Simp[(b*Cos[c + d*x]
)*(b*Csc[c + d*x])^(n - 1))/(d*(n - 1)), x] + Dist[(b^2*(n - 2))/(n - 1), I
nt[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] &&
IntegerQ[2*n]
```

### Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] := -Simp[ArcTanh[Cos[c + d*x]]/d, x]
/; FreeQ[{c, d}, x]
```

### Rubi steps

$$\begin{aligned} \int \csc^3(c + dx)(a - a \csc(c + dx))(A + A \csc(c + dx)) dx &= - \left( (aA) \int \cot^2(c + dx) \csc^3(c + dx) dx \right) \\ &= \frac{aA \cot(c + dx) \csc^3(c + dx)}{4d} + \frac{1}{4}(aA) \int \csc^3(c + dx) dx \\ &= -\frac{aA \cot(c + dx) \csc(c + dx)}{8d} + \frac{aA \cot(c + dx) \csc^3(c + dx)}{4d} \\ &= -\frac{aA \tanh^{-1}(\cos(c + dx))}{8d} - \frac{aA \cot(c + dx) \csc(c + dx)}{8d} + \frac{aA \cot(c + dx) \csc^3(c + dx)}{4d} \end{aligned}$$

**Mathematica [A]** time = 0.044857, size = 117, normalized size = 1.92

$$-aA \left( -\frac{\csc^4\left(\frac{1}{2}(c + dx)\right)}{64d} + \frac{\csc^2\left(\frac{1}{2}(c + dx)\right)}{32d} + \frac{\sec^4\left(\frac{1}{2}(c + dx)\right)}{64d} - \frac{\sec^2\left(\frac{1}{2}(c + dx)\right)}{32d} - \frac{\log\left(\sin\left(\frac{1}{2}(c + dx)\right)\right)}{8d} + \frac{\log\left(\cos\left(\frac{1}{2}(c + dx)\right)\right)}{8d} \right)$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[c + d*x]^3*(a - a*Csc[c + d*x])*(A + A*Csc[c + d*x]),x]
```

```
[Out] -(a*A*(Csc[(c + d*x)/2]^2/(32*d) - Csc[(c + d*x)/2]^4/(64*d) + Log[Cos[(c +
d*x)/2]]/(8*d) - Log[Sin[(c + d*x)/2]]/(8*d) - Sec[(c + d*x)/2]^2/(32*d) +
Sec[(c + d*x)/2]^4/(64*d)))
```

**Maple [A]** time = 0.026, size = 65, normalized size = 1.1

$$-\frac{Aa \cot(dx + c) \csc(dx + c)}{8d} + \frac{Aa \ln(\csc(dx + c) - \cot(dx + c))}{8d} + \frac{Aa \cot(dx + c) (\csc(dx + c))^3}{4d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `int(csc(d*x+c)^3*(a-a*csc(d*x+c))*(A+A*csc(d*x+c)),x)`

[Out]  $-1/8*a*A*\cot(d*x+c)*\csc(d*x+c)/d+1/8/d*A*a*\ln(\csc(d*x+c)-\cot(d*x+c))+1/4*a*A*\cot(d*x+c)*\csc(d*x+c)^3/d$

**Maxima [B]** time = 1.01409, size = 162, normalized size = 2.66

$$\frac{Aa\left(\frac{2(3\cos(dx+c)^3-5\cos(dx+c))}{\cos(dx+c)^4-2\cos(dx+c)^2+1} - 3\log(\cos(dx+c)+1) + 3\log(\cos(dx+c)-1)\right) - 4Aa\left(\frac{2\cos(dx+c)}{\cos(dx+c)^2-1} - \log(\cos(dx+c)+1) + \log(\cos(dx+c)-1)\right)}{16d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate(csc(d*x+c)^3*(a-a*csc(d*x+c))*(A+A*csc(d*x+c)),x, algorithm="maxima")`

[Out]  $-1/16*(A*a*(2*(3*\cos(d*x+c)^3-5*\cos(d*x+c))/(\cos(d*x+c)^4-2*\cos(d*x+c)^2+1)-3*\log(\cos(d*x+c)+1)+3*\log(\cos(d*x+c)-1))-4*A*a*(2*\cos(d*x+c)/(\cos(d*x+c)^2-1)-\log(\cos(d*x+c)+1)+\log(\cos(d*x+c)-1)))/d$

**Fricas [B]** time = 0.496599, size = 344, normalized size = 5.64

$$\frac{2Aa\cos(dx+c)^3+2Aa\cos(dx+c)-\left(Aa\cos(dx+c)^4-2Aa\cos(dx+c)^2+Aa\right)\log\left(\frac{1}{2}\cos(dx+c)+\frac{1}{2}\right)+\left(Aa\cos(dx+c)^4-2Aa\cos(dx+c)^2+Aa\right)\log\left(\frac{1}{2}\cos(dx+c)-\frac{1}{2}\right)}{16\left(d\cos(dx+c)^4-2d\cos(dx+c)^2+d\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate(csc(d*x+c)^3*(a-a*csc(d*x+c))*(A+A*csc(d*x+c)),x, algorithm="fricas")`

[Out]  $1/16*(2*A*a*\cos(d*x+c)^3+2*A*a*\cos(d*x+c)-(A*a*\cos(d*x+c)^4-2*A*a*\cos(d*x+c)^2+A*a)*\log(1/2*\cos(d*x+c)+1/2)+(A*a*\cos(d*x+c)^4-2*A*a*\cos(d*x+c)^2+A*a)*\log(-1/2*\cos(d*x+c)+1/2))/(d*\cos(d*x+c)^4-2*d*\cos(d*x+c)^2+d)$

---

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$-Aa \left( \int -\csc^3(c + dx) dx + \int \csc^5(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*\*3\*(a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x)

[Out] -A\*a\*(Integral(-csc(c + d\*x)\*\*3, x) + Integral(csc(c + d\*x)\*\*5, x))

---

**Giac [A]** time = 1.38111, size = 144, normalized size = 2.36

$$\frac{4 A a \log \left( \frac{|-\cos(dx+c)+1|}{|\cos(dx+c)+1|} \right) - \frac{A a (\cos(dx+c)-1)^2}{(\cos(dx+c)+1)^2} + \frac{\left( A a - \frac{2 A a (\cos(dx+c)-1)^2}{(\cos(dx+c)+1)^2} \right) (\cos(dx+c)+1)^2}{(\cos(dx+c)-1)^2}}{64 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^3\*(a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="giac")

[Out] 1/64\*(4\*A\*a\*log(abs(-cos(d\*x + c) + 1)/abs(cos(d\*x + c) + 1)) - A\*a\*(cos(d\*x + c) - 1)^2/(cos(d\*x + c) + 1)^2 + (A\*a - 2\*A\*a\*(cos(d\*x + c) - 1)^2/(cos(d\*x + c) + 1)^2)\*(cos(d\*x + c) + 1)^2/(cos(d\*x + c) - 1)^2)/d

### 3.8 $\int \csc^2(c+dx)(a-a \csc(c+dx))(A+A \csc(c+dx)) dx$

Optimal. Leaf size=17

$$\frac{aA \cot^3(c+dx)}{3d}$$

[Out] (a\*A\*Cot[c + d\*x]^3)/(3\*d)

**Rubi [A]** time = 0.0724796, antiderivative size = 17, normalized size of antiderivative = 1., number of steps used = 3, number of rules used = 3, integrand size = 30,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.1$ , Rules used = {3962, 2607, 30}

$$\frac{aA \cot^3(c+dx)}{3d}$$

Antiderivative was successfully verified.

[In] Int[Csc[c + d\*x]^2\*(a - a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x]),x]

[Out] (a\*A\*Cot[c + d\*x]^3)/(3\*d)

#### Rule 3962

```
Int[(csc[(e_.) + (f_.)*(x_.)]*(g_.))^(p_.)*(csc[(e_.) + (f_.)*(x_.)]*(b_.) +
(a_.)^(m_.)*(csc[(e_.) + (f_.)*(x_.)]*(d_.) + (c_.))^(n_.), x_Symbol] :> Dist
[(-a*c)^(m), Int[ExpandTrig[(g*csc[e + f*x])^p*cot[e + f*x]^(2*m), (c + d*c
sc[e + f*x])^(n - m), x], x] /; FreeQ[{a, b, c, d, e, f, g, n, p}, x] &
& EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && IntegersQ[m, n] && GeQ[n - m, 0
] && GtQ[m*n, 0]
```

#### Rule 2607

```
Int[sec[(e_.) + (f_.)*(x_.)]^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_.)]^(n_.), x_S
ymbol] :> Dist[1/f, Subst[Int[(b*x)^n*(1 + x^2)^(m/2 - 1), x], x, Tan[e + f
*x]], x] /; FreeQ[{b, e, f, n}, x] && IntegerQ[m/2] && !(IntegerQ[(n - 1)/
2] && LtQ[0, n, m - 1])
```

#### Rule 30

```
Int[(x_)^(m_.), x_Symbol] :> Simp[x^(m + 1)/(m + 1), x] /; FreeQ[m, x] && N
eQ[m, -1]
```

Rubi steps

$$\begin{aligned} \int \csc^2(c + dx)(a - a \csc(c + dx))(A + A \csc(c + dx)) dx &= -\left((aA) \int \cot^2(c + dx) \csc^2(c + dx) dx\right) \\ &= -\frac{(aA) \text{Subst}\left(\int x^2 dx, x, -\cot(c + dx)\right)}{d} \\ &= \frac{aA \cot^3(c + dx)}{3d} \end{aligned}$$

**Mathematica [A]** time = 0.0163463, size = 17, normalized size = 1.

$$\frac{aA \cot^3(c + dx)}{3d}$$

Antiderivative was successfully verified.

[In] Integrate[Csc[c + d\*x]^2\*(a - a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x]),x]

[Out] (a\*A\*Cot[c + d\*x]^3)/(3\*d)

**Maple [B]** time = 0.02, size = 38, normalized size = 2.2

$$\frac{1}{d} \left( -Aa \cot(dx + c) - Aa \left( -\frac{2}{3} - \frac{(\csc(dx + c))^2}{3} \right) \cot(dx + c) \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int(csc(d\*x+c)^2\*(a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x)

[Out] 1/d\*(-A\*a\*cot(d\*x+c)-A\*a\*(-2/3-1/3\*csc(d\*x+c)^2)\*cot(d\*x+c))

**Maxima [B]** time = 1.05654, size = 57, normalized size = 3.35

$$\frac{\frac{3Aa}{\tan(dx+c)} - \frac{(3 \tan(dx+c)^2+1)Aa}{\tan(dx+c)^3}}{3d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^2\*(a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="maxima")

[Out]  $-1/3*(3*A*a/\tan(dx + c) - (3*\tan(dx + c)^2 + 1)*A*a/\tan(dx + c)^3)/d$

**Fricas [B]** time = 0.455756, size = 85, normalized size = 5.

$$\frac{Aa \cos(dx + c)^3}{3(d \cos(dx + c)^2 - d) \sin(dx + c)}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^2\*(a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="fricas")

[Out]  $-1/3*A*a*\cos(dx + c)^3/((d*\cos(dx + c)^2 - d)*\sin(dx + c))$

**Sympy [A]** time = 2.1531, size = 54, normalized size = 3.18

$$\begin{cases} \frac{Aa\left(-\frac{\cot^3(c+dx)}{3}-\cot(c+dx)\right)+Aa\cot(c+dx)}{d} & \text{for } d \neq 0 \\ x(A\csc(c)+A)(-a\csc(c)+a)\csc^2(c) & \text{otherwise} \end{cases}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*\*2\*(a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x)

[Out] Piecewise((- (A\*a\*(-cot(c + d\*x)\*\*3/3 - cot(c + d\*x)) + A\*a\*cot(c + d\*x))/d, Ne(d, 0)), (x\*(A\*csc(c) + A)\*(-a\*csc(c) + a)\*csc(c)\*\*2, True))

**Giac [A]** time = 1.36653, size = 20, normalized size = 1.18

$$\frac{Aa}{3d \tan(dx + c)^3}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^2*(a-a*csc(d*x+c))*(A+A*csc(d*x+c)),x, algorithm="giac")
```

```
[Out] 1/3*A*a/(d*tan(d*x + c)^3)
```

### 3.9 $\int \csc(c+dx)(a-a \csc(c+dx))(A+A \csc(c+dx)) dx$

**Optimal.** Leaf size=38

$$\frac{aA \cot(c+dx) \csc(c+dx)}{2d} - \frac{aA \tanh^{-1}(\cos(c+dx))}{2d}$$

[Out]  $-(a*A*ArcTanh[Cos[c + d*x]])/(2*d) + (a*A*Cot[c + d*x]*Csc[c + d*x])/(2*d)$

**Rubi [A]** time = 0.0509353, antiderivative size = 38, normalized size of antiderivative = 1., number of steps used = 3, number of rules used = 3, integrand size = 28,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.107$ , Rules used = {3958, 2611, 3770}

$$\frac{aA \cot(c+dx) \csc(c+dx)}{2d} - \frac{aA \tanh^{-1}(\cos(c+dx))}{2d}$$

Antiderivative was successfully verified.

[In]  $\text{Int}[Csc[c + d*x]*(a - a*Csc[c + d*x])*(A + A*Csc[c + d*x]),x]$

[Out]  $-(a*A*ArcTanh[Cos[c + d*x]])/(2*d) + (a*A*Cot[c + d*x]*Csc[c + d*x])/(2*d)$

#### Rule 3958

$\text{Int}[csc[(e_.) + (f_.)*(x_.)]*(csc[(e_.) + (f_.)*(x_.)]*(b_.) + (a_.))^{(m_.)}*(csc[(e_.) + (f_.)*(x_.)]*(d_.) + (c_.))^{(n_.)}, x\_Symbol] \rightarrow \text{Dist}[(-a*c)^m, \text{Int}[\text{ExpandTrig}[csc[e + f*x]*cot[e + f*x]^{(2*m)}, (c + d*csc[e + f*x])^{(n - m)}, x], x] /; \text{FreeQ}\{a, b, c, d, e, f, n\}, x] \&\& \text{EqQ}[b*c + a*d, 0] \&\& \text{EqQ}[a^2 - b^2, 0] \&\& \text{IntegersQ}[m, n] \&\& \text{GeQ}[n - m, 0] \&\& \text{GtQ}[m*n, 0]$

#### Rule 2611

$\text{Int}[(a_.)*sec[(e_.) + (f_.)*(x_.)]^{(m_.)}*((b_.)*tan[(e_.) + (f_.)*(x_.)]^{(n_.)}, x\_Symbol] \rightarrow \text{Simp}[(b*(a*Sec[e + f*x])^m*(b*Tan[e + f*x])^{(n - 1)})/(f*(m + n - 1)), x] - \text{Dist}[(b^2*(n - 1))/(m + n - 1), \text{Int}[(a*Sec[e + f*x])^m*(b*Tan[e + f*x])^{(n - 2)}, x], x] /; \text{FreeQ}\{a, b, e, f, m\}, x] \&\& \text{GtQ}[n, 1] \&\& \text{NeQ}[m + n - 1, 0] \&\& \text{IntegersQ}[2*m, 2*n]$

#### Rule 3770

$\text{Int}[csc[(c_.) + (d_.)*(x_.)], x\_Symbol] \rightarrow -\text{Simp}[ArcTanh[Cos[c + d*x]]/d, x] /; \text{FreeQ}\{c, d\}, x]$



Rubi steps

$$\begin{aligned} \int \csc(c + dx)(a - a \csc(c + dx))(A + A \csc(c + dx)) dx &= -\left( (aA) \int \cot^2(c + dx) \csc(c + dx) dx \right) \\ &= \frac{aA \cot(c + dx) \csc(c + dx)}{2d} + \frac{1}{2}(aA) \int \csc(c + dx) dx \\ &= -\frac{aA \tanh^{-1}(\cos(c + dx))}{2d} + \frac{aA \cot(c + dx) \csc(c + dx)}{2d} \end{aligned}$$

**Mathematica [B]** time = 0.03455, size = 79, normalized size = 2.08

$$-aA \left( -\frac{\csc^2\left(\frac{1}{2}(c + dx)\right)}{8d} + \frac{\sec^2\left(\frac{1}{2}(c + dx)\right)}{8d} - \frac{\log\left(\sin\left(\frac{1}{2}(c + dx)\right)\right)}{2d} + \frac{\log\left(\cos\left(\frac{1}{2}(c + dx)\right)\right)}{2d} \right)$$

Antiderivative was successfully verified.

[In] Integrate[Csc[c + d\*x]\*(a - a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x]),x]

[Out] -(a\*A\*(-Csc[(c + d\*x)/2]^2/(8\*d) + Log[Cos[(c + d\*x)/2]]/(2\*d) - Log[Sin[(c + d\*x)/2]]/(2\*d) + Sec[(c + d\*x)/2]^2/(8\*d)))

**Maple [A]** time = 0.023, size = 44, normalized size = 1.2

$$\frac{Aa \ln(\csc(dx + c) - \cot(dx + c))}{2d} + \frac{Aa \cot(dx + c) \csc(dx + c)}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int(csc(d\*x+c)\*(a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x)

[Out] 1/2/d\*A\*a\*ln(csc(d\*x+c)-cot(d\*x+c))+1/2\*a\*A\*cot(d\*x+c)\*csc(d\*x+c)/d

**Maxima [A]** time = 1.11618, size = 92, normalized size = 2.42

$$\frac{Aa \left( \frac{2 \cos(dx+c)}{\cos(dx+c)^2-1} - \log(\cos(dx+c)+1) + \log(\cos(dx+c)-1) \right) + 4 Aa \log(\cot(dx+c) + \csc(dx+c))}{4d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="maxima")

[Out]  $-1/4*(A*a*(2*\cos(d*x + c)/(\cos(d*x + c)^2 - 1) - \log(\cos(d*x + c) + 1) + \log(\cos(d*x + c) - 1)) + 4*A*a*\log(\cot(d*x + c) + \csc(d*x + c)))/d$

**Fricas [B]** time = 0.495874, size = 223, normalized size = 5.87

$$\frac{2 A a \cos(dx + c) + (A a \cos(dx + c)^2 - A a) \log\left(\frac{1}{2} \cos(dx + c) + \frac{1}{2}\right) - (A a \cos(dx + c)^2 - A a) \log\left(-\frac{1}{2} \cos(dx + c) + \frac{1}{2}\right)}{4(d \cos(dx + c)^2 - d)}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x, algorithm="fricas")

[Out]  $-1/4*(2*A*a*\cos(d*x + c) + (A*a*\cos(d*x + c)^2 - A*a)*\log(1/2*\cos(d*x + c) + 1/2) - (A*a*\cos(d*x + c)^2 - A*a)*\log(-1/2*\cos(d*x + c) + 1/2))/(d*\cos(d*x + c)^2 - d)$

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$-Aa \left( \int -\csc(c + dx) dx + \int \csc^3(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c)),x)

[Out]  $-A*a*(\text{Integral}(-\csc(c + d*x), x) + \text{Integral}(\csc(c + d*x)**3, x))$

**Giac [B]** time = 1.33333, size = 136, normalized size = 3.58

$$\frac{2 A a \log\left(\frac{|-\cos(dx+c)+1|}{|\cos(dx+c)+1|}\right) + \frac{A a (\cos(dx+c)-1)}{\cos(dx+c)+1} - \frac{\left(A a + \frac{2 A a (\cos(dx+c)-1)}{\cos(dx+c)+1}\right) (\cos(dx+c)+1)}{\cos(dx+c)-1}}{8 d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)*(a-a*csc(d*x+c))*(A+A*csc(d*x+c)),x, algorithm="giac")
```

```
[Out] 1/8*(2*A*a*log(abs(-cos(d*x + c) + 1)/abs(cos(d*x + c) + 1)) + A*a*(cos(d*x + c) - 1)/(cos(d*x + c) + 1) - (A*a + 2*A*a*(cos(d*x + c) - 1)/(cos(d*x + c) + 1))*(cos(d*x + c) + 1)/(cos(d*x + c) - 1))/d
```

### 3.10 $\int (a - a \csc(c + dx))(A + A \csc(c + dx)) \sin(c + dx) dx$

**Optimal.** Leaf size=27

$$\frac{aA \tanh^{-1}(\cos(c + dx))}{d} - \frac{aA \cos(c + dx)}{d}$$

[Out] (a\*A\*ArcTanh[Cos[c + d\*x]])/d - (a\*A\*Cos[c + d\*x])/d

**Rubi [A]** time = 0.0529919, antiderivative size = 27, normalized size of antiderivative = 1., number of steps used = 4, number of rules used = 4, integrand size = 28,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$ , Rules used = {3962, 2592, 321, 206}

$$\frac{aA \tanh^{-1}(\cos(c + dx))}{d} - \frac{aA \cos(c + dx)}{d}$$

Antiderivative was successfully verified.

[In] Int[(a - a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x])\*Sin[c + d\*x], x]

[Out] (a\*A\*ArcTanh[Cos[c + d\*x]])/d - (a\*A\*Cos[c + d\*x])/d

#### Rule 3962

Int[(csc[(e\_.) + (f\_.)\*(x\_.)]\*(g\_.))^(p\_.)\*(csc[(e\_.) + (f\_.)\*(x\_.)]\*(b\_.) + (a\_.)^(m\_.)\*(csc[(e\_.) + (f\_.)\*(x\_.)]\*(d\_.) + (c\_.)^(n\_.), x\_Symbol] := Dist[(-a\*c)^m, Int[ExpandTrig[(g\*csc[e + f\*x])^p\*cot[e + f\*x]^(2\*m), (c + d\*csc[e + f\*x])^(n - m), x], x], x] /; FreeQ[{a, b, c, d, e, f, g, n, p}, x] && EqQ[b\*c + a\*d, 0] && EqQ[a^2 - b^2, 0] && IntegersQ[m, n] && GeQ[n - m, 0] && GtQ[m\*n, 0]

#### Rule 2592

Int[((a\_.)\*sin[(e\_.) + (f\_.)\*(x\_.)])^(m\_.)\*tan[(e\_.) + (f\_.)\*(x\_.)]^(n\_.), x\_Symbol] := With[{ff = FreeFactors[Sin[e + f\*x], x]}, Dist[ff/f, Subst[Int[(ff\*x)^(m + n)/(a^2 - ff^2\*x^2)^((n + 1)/2), x], x, (a\*Sin[e + f\*x])/ff], x] /; FreeQ[{a, e, f, m}, x] && IntegerQ[(n + 1)/2]

#### Rule 321

Int[((c\_.)\*(x\_.))^(m\_.)\*((a\_.) + (b\_.)\*(x\_.)^(n\_.))^(p\_.), x\_Symbol] := Simp[(c^(n - 1)\*(c\*x)^(m - n + 1)\*(a + b\*x^n)^(p + 1))/(b\*(m + n\*p + 1)), x] - Dist[

$(a*c^n*(m - n + 1))/(b*(m + n*p + 1)), \text{Int}[(c*x)^(m - n)*(a + b*x^n)^p, x],$   
 $x] /; \text{FreeQ}\{a, b, c, p\}, x \} \&\& \text{IGtQ}[n, 0] \&\& \text{GtQ}[m, n - 1] \&\& \text{NeQ}[m + n*p$   
 $+ 1, 0] \&\& \text{IntBinomialQ}[a, b, c, n, m, p, x]$

### Rule 206

$\text{Int}[(a_ + (b_.)*(x_)^2)^{-1}, x\_Symbol] \rightarrow \text{Simp}[(1*\text{ArcTanh}[(\text{Rt}[-b, 2]*x)/$   
 $\text{Rt}[a, 2]])/(\text{Rt}[a, 2]*\text{Rt}[-b, 2]), x] /; \text{FreeQ}\{a, b\}, x \} \&\& \text{NegQ}[a/b] \&\& (\text{Gt}$   
 $\text{Q}[a, 0] \ || \ \text{LtQ}[b, 0])$

### Rubi steps

$$\begin{aligned} \int (a - a \csc(c + dx))(A + A \csc(c + dx)) \sin(c + dx) dx &= -((aA) \int \cos(c + dx) \cot(c + dx) dx) \\ &= \frac{(aA) \text{Subst} \left( \int \frac{x^2}{1-x^2} dx, x, \cos(c + dx) \right)}{d} \\ &= -\frac{aA \cos(c + dx)}{d} + \frac{(aA) \text{Subst} \left( \int \frac{1}{1-x^2} dx, x, \cos(c + dx) \right)}{d} \\ &= \frac{aA \tanh^{-1}(\cos(c + dx))}{d} - \frac{aA \cos(c + dx)}{d} \end{aligned}$$

**Mathematica [A]** time = 0.0244733, size = 46, normalized size = 1.7

$$-aA \left( \frac{\cos(c + dx)}{d} + \frac{\log \left( \sin \left( \frac{1}{2}(c + dx) \right) \right)}{d} - \frac{\log \left( \cos \left( \frac{1}{2}(c + dx) \right) \right)}{d} \right)$$

Antiderivative was successfully verified.

[In] Integrate[(a - a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x])\*Sin[c + d\*x], x]

[Out] -(a\*A\*(Cos[c + d\*x]/d - Log[Cos[(c + d\*x)/2]]/d + Log[Sin[(c + d\*x)/2]]/d))

**Maple [A]** time = 0.041, size = 38, normalized size = 1.4

$$-\frac{Aa \cos(dx + c)}{d} - \frac{Aa \ln(\csc(dx + c) - \cot(dx + c))}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `int((a-a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c),x)`

[Out] `-a*A*cos(d*x+c)/d-1/d*A*a*ln(csc(d*x+c)-cot(d*x+c))`

**Maxima [A]** time = 1.1122, size = 54, normalized size = 2.

$$\frac{Aa(\log(\cos(dx+c)+1) - \log(\cos(dx+c)-1)) - 2Aa\cos(dx+c)}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a-a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c),x, algorithm="maxima")`

[Out] `1/2*(A*a*(log(cos(d*x+c)+1) - log(cos(d*x+c)-1)) - 2*A*a*cos(d*x+c))/d`

**Fricas [A]** time = 0.500541, size = 132, normalized size = 4.89

$$\frac{2Aa\cos(dx+c) - Aa\log\left(\frac{1}{2}\cos(dx+c) + \frac{1}{2}\right) + Aa\log\left(-\frac{1}{2}\cos(dx+c) + \frac{1}{2}\right)}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a-a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c),x, algorithm="fricas")`

[Out] `-1/2*(2*A*a*cos(d*x+c) - A*a*log(1/2*cos(d*x+c) + 1/2) + A*a*log(-1/2*cos(d*x+c) + 1/2))/d`

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$-Aa\left(\int \sin(c+dx)\csc^2(c+dx)dx + \int -\sin(c+dx)dx\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a-a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c),x)
```

```
[Out] -A*a*(Integral(sin(c + d*x)*csc(c + d*x)**2, x) + Integral(-sin(c + d*x), x
))
```

**Giac [B]** time = 1.36561, size = 81, normalized size = 3.

$$\frac{Aa \log\left(\frac{|-\cos(dx+c)+1|}{|\cos(dx+c)+1|}\right) - \frac{4Aa}{\frac{\cos(dx+c)-1}{\cos(dx+c)+1} - 1}}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a-a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c),x, algorithm="giac")
```

```
[Out] -1/2*(A*a*log(abs(-cos(d*x + c) + 1)/abs(cos(d*x + c) + 1)) - 4*A*a/((cos(d
*x + c) - 1)/(cos(d*x + c) + 1) - 1))/d
```

### 3.11 $\int (a - a \csc(c + dx))(A + A \csc(c + dx)) \sin^2(c + dx) dx$

**Optimal.** Leaf size=29

$$-\frac{aA \sin(c + dx) \cos(c + dx)}{2d} - \frac{1}{2}aAx$$

[Out]  $-(aAx)/2 - (aA \cos[c + dx] \sin[c + dx])/(2d)$

**Rubi [A]** time = 0.0549121, antiderivative size = 29, normalized size of antiderivative = 1., number of steps used = 3, number of rules used = 3, integrand size = 30,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.1$ , Rules used = {3962, 2635, 8}

$$-\frac{aA \sin(c + dx) \cos(c + dx)}{2d} - \frac{1}{2}aAx$$

Antiderivative was successfully verified.

[In]  $\text{Int}[(a - a \csc[c + dx])(A + A \csc[c + dx]) \sin^2[c + dx], x]$

[Out]  $-(aAx)/2 - (aA \cos[c + dx] \sin[c + dx])/(2d)$

#### Rule 3962

```
Int[(csc[(e_.) + (f_.)*(x_.)]*(g_.))^(p_.)*(csc[(e_.) + (f_.)*(x_.)]*(b_.) +
(a_.)^(m_.)*(csc[(e_.) + (f_.)*(x_.)]*(d_.) + (c_.)^(n_.), x_Symbol] := Dist
[(-a*c)^(m), Int[ExpandTrig[(g*csc[e + f*x])^p*cot[e + f*x]^(2*m), (c + d*c
sc[e + f*x])^(n - m), x], x], x] /; FreeQ[{a, b, c, d, e, f, g, n, p}, x] &
& EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && IntegersQ[m, n] && GeQ[n - m, 0
] && GtQ[m*n, 0]
```

#### Rule 2635

```
Int[((b_.)*sin[(c_.) + (d_.)*(x_.)])^(n_.), x_Symbol] := -Simp[(b*cos[c + d*x]
)*(b*sin[c + d*x])^(n - 1)/(d*n), x] + Dist[(b^2*(n - 1))/n, Int[(b*sin[c
+ d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n
]
```

#### Rule 8

```
Int[a_, x_Symbol] := Simp[a*x, x] /; FreeQ[a, x]
```



Rubi steps

$$\begin{aligned} \int (a - a \csc(c + dx))(A + A \csc(c + dx)) \sin^2(c + dx) dx &= - \left( (aA) \int \cos^2(c + dx) dx \right) \\ &= - \frac{aA \cos(c + dx) \sin(c + dx)}{2d} - \frac{1}{2}(aA) \int 1 dx \\ &= -\frac{1}{2}aAx - \frac{aA \cos(c + dx) \sin(c + dx)}{2d} \end{aligned}$$

**Mathematica [A]** time = 0.022163, size = 25, normalized size = 0.86

$$\frac{aA(2(c + dx) + \sin(2(c + dx)))}{4d}$$

Antiderivative was successfully verified.

[In] Integrate[(a - a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x])\*Sin[c + d\*x]^2,x]

[Out] -(a\*A\*(2\*(c + d\*x) + Sin[2\*(c + d\*x)]))/(4\*d)

**Maple [A]** time = 0.036, size = 40, normalized size = 1.4

$$\frac{1}{d} \left( Aa \left( -\frac{\cos(dx + c) \sin(dx + c)}{2} + \frac{dx}{2} + \frac{c}{2} \right) - Aa(dx + c) \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int((a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c))\*sin(d\*x+c)^2,x)

[Out] 1/d\*(A\*a\*(-1/2\*cos(d\*x+c)\*sin(d\*x+c)+1/2\*d\*x+1/2\*c)-A\*a\*(d\*x+c))

**Maxima [A]** time = 1.03968, size = 50, normalized size = 1.72

$$\frac{(2 dx + 2 c - \sin(2 dx + 2 c))Aa - 4 (dx + c)Aa}{4 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c))\*sin(d\*x+c)^2,x, algorithm="maxima")

[Out]  $\frac{1}{4} * ((2*d*x + 2*c - \sin(2*d*x + 2*c)) * A*a - 4*(d*x + c) * A*a) / d$

**Fricas [A]** time = 0.475762, size = 68, normalized size = 2.34

$$-\frac{Aadx + Aa \cos(dx + c) \sin(dx + c)}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c))\*sin(d\*x+c)^2,x, algorithm="fricas")

[Out]  $-1/2 * (A*a*d*x + A*a*\cos(d*x + c) * \sin(d*x + c)) / d$

**Sympy [A]** time = 13.3814, size = 54, normalized size = 1.86

$$-Aax + Aa \left\{ \begin{array}{ll} \frac{x \sin^2(c+dx)}{2} + \frac{x \cos^2(c+dx)}{2} - \frac{\sin(c+dx) \cos(c+dx)}{2d} & \text{for } d \neq 0 \\ x \sin^2(c) & \text{otherwise} \end{array} \right.$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c))\*sin(d\*x+c)\*\*2,x)

[Out]  $-A*a*x + A*a*\text{Piecewise}((x*\sin(c + d*x)**2/2 + x*\cos(c + d*x)**2/2 - \sin(c + d*x)*\cos(c + d*x)/(2*d), \text{Ne}(d, 0)), (x*\sin(c)**2, \text{True}))$

**Giac [A]** time = 1.25946, size = 47, normalized size = 1.62

$$-\frac{(dx + c)Aa + \frac{Aa \tan(dx+c)}{\tan(dx+c)^2+1}}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a-a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c)^2,x, algorithm="giac")
```

```
[Out] -1/2*((d*x + c)*A*a + A*a*tan(d*x + c)/(tan(d*x + c)^2 + 1))/d
```

### 3.12 $\int (a - a \csc(c + dx))(A + A \csc(c + dx)) \sin^3(c + dx) dx$

Optimal. Leaf size=17

$$\frac{aA \cos^3(c + dx)}{3d}$$

[Out] (a\*A\*Cos[c + d\*x]^3)/(3\*d)

**Rubi [A]** time = 0.0612404, antiderivative size = 17, normalized size of antiderivative = 1., number of steps used = 3, number of rules used = 3, integrand size = 30,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.1$ , Rules used = {3962, 2565, 30}

$$\frac{aA \cos^3(c + dx)}{3d}$$

Antiderivative was successfully verified.

[In] Int[(a - a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x])\*Sin[c + d\*x]^3,x]

[Out] (a\*A\*Cos[c + d\*x]^3)/(3\*d)

#### Rule 3962

```
Int[(csc[(e_.) + (f_.)*(x_.)]*(g_.))^(p_.)*(csc[(e_.) + (f_.)*(x_.)]*(b_.) +
(a_.)^(m_.)*(csc[(e_.) + (f_.)*(x_.)]*(d_.) + (c_.))^(n_.), x_Symbol] :> Dist
[(-a*c)^(m), Int[ExpandTrig[(g*csc[e + f*x])^p*cot[e + f*x]^(2*m), (c + d*c
sc[e + f*x])^(n - m), x], x] /; FreeQ[{a, b, c, d, e, f, g, n, p}, x] &
& EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && IntegersQ[m, n] && GeQ[n - m, 0
] && GtQ[m*n, 0]
```

#### Rule 2565

```
Int[(cos[(e_.) + (f_.)*(x_.)]*(a_.))^(m_.)*sin[(e_.) + (f_.)*(x_.)]^(n_.), x_
Symbol] :> -Dist[(a*f)^(-1), Subst[Int[x^m*(1 - x^2/a^2)^((n - 1)/2), x], x
, a*Cos[e + f*x]], x] /; FreeQ[{a, e, f, m}, x] && IntegerQ[(n - 1)/2] &&
!(IntegerQ[(m - 1)/2] && GtQ[m, 0] && LeQ[m, n])
```

#### Rule 30

```
Int[(x_)^(m_.), x_Symbol] :> Simp[x^(m + 1)/(m + 1), x] /; FreeQ[m, x] && N
eQ[m, -1]
```

Rubi steps

$$\begin{aligned} \int (a - a \csc(c + dx))(A + A \csc(c + dx)) \sin^3(c + dx) dx &= -\left((aA) \int \cos^2(c + dx) \sin(c + dx) dx\right) \\ &= \frac{(aA) \text{Subst}\left(\int x^2 dx, x, \cos(c + dx)\right)}{d} \\ &= \frac{aA \cos^3(c + dx)}{3d} \end{aligned}$$

**Mathematica [A]** time = 0.0071892, size = 17, normalized size = 1.

$$\frac{aA \cos^3(c + dx)}{3d}$$

Antiderivative was successfully verified.

[In] Integrate[(a - a\*Csc[c + d\*x])\*(A + A\*Csc[c + d\*x])\*Sin[c + d\*x]^3,x]

[Out] (a\*A\*Cos[c + d\*x]^3)/(3\*d)

**Maple [B]** time = 0.036, size = 35, normalized size = 2.1

$$\frac{1}{d} \left( -\frac{Aa(2 + (\sin(dx + c))^2) \cos(dx + c)}{3} + Aa \cos(dx + c) \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int((a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c))\*sin(d\*x+c)^3,x)

[Out] 1/d\*(-1/3\*A\*a\*(2+sin(d\*x+c)^2)\*cos(d\*x+c)+A\*a\*cos(d\*x+c))

**Maxima [B]** time = 1.07665, size = 49, normalized size = 2.88

$$\frac{(\cos(dx + c)^3 - 3 \cos(dx + c))Aa + 3 Aa \cos(dx + c)}{3d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c))\*sin(d\*x+c)^3,x, algorithm="maxima")

[Out] 1/3\*((cos(d\*x + c)^3 - 3\*cos(d\*x + c))\*A\*a + 3\*A\*a\*cos(d\*x + c))/d

**Fricas [A]** time = 0.471494, size = 35, normalized size = 2.06

$$\frac{Aa \cos(dx + c)^3}{3d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c))\*sin(d\*x+c)^3,x, algorithm="fricas")

[Out] 1/3\*A\*a\*cos(d\*x + c)^3/d

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$-Aa \left( \int \sin^3(c + dx) \csc^2(c + dx) dx + \int -\sin^3(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a-a\*csc(d\*x+c))\*(A+A\*csc(d\*x+c))\*sin(d\*x+c)\*\*3,x)

[Out] -A\*a\*(Integral(sin(c + d\*x)\*\*3\*csc(c + d\*x)\*\*2, x) + Integral(-sin(c + d\*x)\*\*3, x))

**Giac [B]** time = 1.35893, size = 76, normalized size = 4.47

$$\frac{2 \left( Aa + \frac{3Aa(\cos(dx+c)-1)^2}{(\cos(dx+c)+1)^2} \right)}{3d \left( \frac{\cos(dx+c)-1}{\cos(dx+c)+1} - 1 \right)^3}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a-a*csc(d*x+c))*(A+A*csc(d*x+c))*sin(d*x+c)^3,x, algorithm="giac")
```

```
[Out] -2/3*(A*a + 3*A*a*(cos(d*x + c) - 1)^2/(cos(d*x + c) + 1)^2)/(d*((cos(d*x + c) - 1)/(cos(d*x + c) + 1) - 1)^3)
```

### 3.13 $\int \csc^3(c + dx)(a + a \csc(c + dx))(A - A \csc(c + dx)) dx$

**Optimal.** Leaf size=61

$$-\frac{aA \tanh^{-1}(\cos(c + dx))}{8d} + \frac{aA \cot(c + dx) \csc^3(c + dx)}{4d} - \frac{aA \cot(c + dx) \csc(c + dx)}{8d}$$

[Out]  $-(a*A*ArcTanh[Cos[c + d*x]])/(8*d) - (a*A*Cot[c + d*x]*Csc[c + d*x])/(8*d) + (a*A*Cot[c + d*x]*Csc[c + d*x]^3)/(4*d)$

**Rubi [A]** time = 0.0902824, antiderivative size = 61, normalized size of antiderivative = 1., number of steps used = 4, number of rules used = 4, integrand size = 30,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.133$ , Rules used = {3962, 2611, 3768, 3770}

$$-\frac{aA \tanh^{-1}(\cos(c + dx))}{8d} + \frac{aA \cot(c + dx) \csc^3(c + dx)}{4d} - \frac{aA \cot(c + dx) \csc(c + dx)}{8d}$$

Antiderivative was successfully verified.

[In]  $\text{Int}[Csc[c + d*x]^3*(a + a*Csc[c + d*x])*(A - A*Csc[c + d*x]), x]$

[Out]  $-(a*A*ArcTanh[Cos[c + d*x]])/(8*d) - (a*A*Cot[c + d*x]*Csc[c + d*x])/(8*d) + (a*A*Cot[c + d*x]*Csc[c + d*x]^3)/(4*d)$

#### Rule 3962

```
Int[(csc[(e_.) + (f_.)*(x_)]*(g_.))^(p_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_.)^(m_.)*(csc[(e_.) + (f_.)*(x_)]*(d_.) + (c_.))^(n_.), x_Symbol] :> Dist
[(-a*c)^(m), Int[ExpandTrig[(g*csc[e + f*x])^p*cot[e + f*x]^(2*m), (c + d*c
sc[e + f*x])^(n - m), x], x], x] /; FreeQ[{a, b, c, d, e, f, g, n, p}, x] &
& EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && IntegersQ[m, n] && GeQ[n - m, 0]
] && GtQ[m*n, 0]
```

#### Rule 2611

```
Int[((a_.)*sec[(e_.) + (f_.)*(x_)])^(m_.)*((b_.)*tan[(e_.) + (f_.)*(x_)])^(
n_), x_Symbol] :> Simp[(b*(a*Sec[e + f*x])^m*(b*Tan[e + f*x])^(n - 1))/(f*(
m + n - 1)), x] - Dist[(b^2*(n - 1))/(m + n - 1), Int[(a*Sec[e + f*x])^m*(b
*Tan[e + f*x])^(n - 2), x], x] /; FreeQ[{a, b, e, f, m}, x] && GtQ[n, 1] &&
NeQ[m + n - 1, 0] && IntegersQ[2*m, 2*n]
```



Rule 3768

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := -Simp[(b*Cos[c + d*x]
]*(b*Csc[c + d*x])^(n - 1))/(d*(n - 1)), x] + Dist[(b^2*(n - 2))/(n - 1), I
nt[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] &&
IntegerQ[2*n]
```

Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] := -Simp[ArcTanh[Cos[c + d*x]]/d, x]
/; FreeQ[{c, d}, x]
```

Rubi steps

$$\begin{aligned} \int \csc^3(c + dx)(a + a \csc(c + dx))(A - A \csc(c + dx)) dx &= - \left( (aA) \int \cot^2(c + dx) \csc^3(c + dx) dx \right) \\ &= \frac{aA \cot(c + dx) \csc^3(c + dx)}{4d} + \frac{1}{4}(aA) \int \csc^3(c + dx) dx \\ &= -\frac{aA \cot(c + dx) \csc(c + dx)}{8d} + \frac{aA \cot(c + dx) \csc^3(c + dx)}{4d} \\ &= -\frac{aA \tanh^{-1}(\cos(c + dx))}{8d} - \frac{aA \cot(c + dx) \csc(c + dx)}{8d} + \frac{aA \cot(c + dx) \csc^3(c + dx)}{4d} \end{aligned}$$

**Mathematica [A]** time = 0.0380465, size = 117, normalized size = 1.92

$$-aA \left( -\frac{\csc^4\left(\frac{1}{2}(c + dx)\right)}{64d} + \frac{\csc^2\left(\frac{1}{2}(c + dx)\right)}{32d} + \frac{\sec^4\left(\frac{1}{2}(c + dx)\right)}{64d} - \frac{\sec^2\left(\frac{1}{2}(c + dx)\right)}{32d} - \frac{\log\left(\sin\left(\frac{1}{2}(c + dx)\right)\right)}{8d} + \frac{\log\left(\cos\left(\frac{1}{2}(c + dx)\right)\right)}{8d} \right)$$

Antiderivative was successfully verified.

```
[In] Integrate[Csc[c + d*x]^3*(a + a*Csc[c + d*x])*(A - A*Csc[c + d*x]),x]
```

```
[Out] -(a*A*(Csc[(c + d*x)/2]^2/(32*d) - Csc[(c + d*x)/2]^4/(64*d) + Log[Cos[(c +
d*x)/2]]/(8*d) - Log[Sin[(c + d*x)/2]]/(8*d) - Sec[(c + d*x)/2]^2/(32*d) +
Sec[(c + d*x)/2]^4/(64*d)))
```

**Maple [A]** time = 0.027, size = 65, normalized size = 1.1

$$-\frac{Aa \cot(dx + c) \csc(dx + c)}{8d} + \frac{Aa \ln(\csc(dx + c) - \cot(dx + c))}{8d} + \frac{Aa \cot(dx + c) (\csc(dx + c))^3}{4d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `int(csc(d*x+c)^3*(a+a*csc(d*x+c))*(A-A*csc(d*x+c)),x)`

[Out]  $-1/8*a*A*\cot(d*x+c)*\csc(d*x+c)/d+1/8/d*A*a*\ln(\csc(d*x+c)-\cot(d*x+c))+1/4*a*A*\cot(d*x+c)*\csc(d*x+c)^3/d$

**Maxima [B]** time = 1.00009, size = 162, normalized size = 2.66

$$\frac{Aa\left(\frac{2(3\cos(dx+c)^3-5\cos(dx+c))}{\cos(dx+c)^4-2\cos(dx+c)^2+1} - 3\log(\cos(dx+c)+1) + 3\log(\cos(dx+c)-1)\right) - 4Aa\left(\frac{2\cos(dx+c)}{\cos(dx+c)^2-1} - \log(\cos(dx+c)+1)\right)}{16d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate(csc(d*x+c)^3*(a+a*csc(d*x+c))*(A-A*csc(d*x+c)),x, algorithm="maxima")`

[Out]  $-1/16*(A*a*(2*(3*\cos(d*x+c)^3-5*\cos(d*x+c))/(\cos(d*x+c)^4-2*\cos(d*x+c)^2+1)-3*\log(\cos(d*x+c)+1)+3*\log(\cos(d*x+c)-1))-4*A*a*(2*\cos(d*x+c)/(\cos(d*x+c)^2-1)-\log(\cos(d*x+c)+1)+\log(\cos(d*x+c)-1)))/d$

**Fricas [B]** time = 0.497152, size = 344, normalized size = 5.64

$$\frac{2Aa\cos(dx+c)^3+2Aa\cos(dx+c)-\left(Aa\cos(dx+c)^4-2Aa\cos(dx+c)^2+Aa\right)\log\left(\frac{1}{2}\cos(dx+c)+\frac{1}{2}\right)+\left(Aa\cos(dx+c)^4-2Aa\cos(dx+c)^2+Aa\right)\log\left(\frac{1}{2}\cos(dx+c)-\frac{1}{2}\right)}{16\left(d\cos(dx+c)^4-2d\cos(dx+c)^2+d\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate(csc(d*x+c)^3*(a+a*csc(d*x+c))*(A-A*csc(d*x+c)),x, algorithm="fricas")`

[Out]  $1/16*(2*A*a*\cos(d*x+c)^3+2*A*a*\cos(d*x+c)-(A*a*\cos(d*x+c)^4-2*A*a*\cos(d*x+c)^2+A*a)*\log(1/2*\cos(d*x+c)+1/2)+(A*a*\cos(d*x+c)^4-2*A*a*\cos(d*x+c)^2+A*a)*\log(-1/2*\cos(d*x+c)+1/2))/(d*\cos(d*x+c)^4-2*d*\cos(d*x+c)^2+d)$

---

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$-Aa \left( \int -\csc^3(c + dx) dx + \int \csc^5(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*\*3\*(a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)), x)

[Out] -A\*a\*(Integral(-csc(c + d\*x)\*\*3, x) + Integral(csc(c + d\*x)\*\*5, x))

---

**Giac [A]** time = 1.42591, size = 144, normalized size = 2.36

$$\frac{4 A a \log \left( \frac{|-\cos(dx+c)+1|}{|\cos(dx+c)+1|} \right) - \frac{A a (\cos(dx+c)-1)^2}{(\cos(dx+c)+1)^2} + \frac{\left( A a - \frac{2 A a (\cos(dx+c)-1)^2}{(\cos(dx+c)+1)^2} \right) (\cos(dx+c)+1)^2}{(\cos(dx+c)-1)^2}}{64 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^3\*(a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)), x, algorithm="giac")

[Out] 1/64\*(4\*A\*a\*log(abs(-cos(d\*x + c) + 1)/abs(cos(d\*x + c) + 1)) - A\*a\*(cos(d\*x + c) - 1)^2/(cos(d\*x + c) + 1)^2 + (A\*a - 2\*A\*a\*(cos(d\*x + c) - 1)^2/(cos(d\*x + c) + 1)^2)\*(cos(d\*x + c) + 1)^2/(cos(d\*x + c) - 1)^2)/d

$$3.14 \quad \int \csc^2(c + dx)(a + a \csc(c + dx))(A - A \csc(c + dx)) dx$$

**Optimal.** Leaf size=17

$$\frac{aA \cot^3(c + dx)}{3d}$$

[Out] (a\*A\*Cot[c + d\*x]^3)/(3\*d)

**Rubi [A]** time = 0.0710912, antiderivative size = 17, normalized size of antiderivative = 1., number of steps used = 3, number of rules used = 3, integrand size = 30,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.1$ , Rules used = {3962, 2607, 30}

$$\frac{aA \cot^3(c + dx)}{3d}$$

Antiderivative was successfully verified.

[In] Int[Csc[c + d\*x]^2\*(a + a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x]),x]

[Out] (a\*A\*Cot[c + d\*x]^3)/(3\*d)

#### Rule 3962

Int[(csc[(e\_.) + (f\_.)\*(x\_.)]\*(g\_.))^p\_.\*(csc[(e\_.) + (f\_.)\*(x\_.)]\*(b\_.) + (a\_.))^m\_.\*(csc[(e\_.) + (f\_.)\*(x\_.)]\*(d\_.) + (c\_.))^n\_.), x\_Symbol] :> Dist[(-a\*c)^m, Int[ExpandTrig[(g\*csc[e + f\*x])^p\*cot[e + f\*x]^(2\*m), (c + d\*csc[e + f\*x])^(n - m), x], x], x] /; FreeQ[{a, b, c, d, e, f, g, n, p}, x] && EqQ[b\*c + a\*d, 0] && EqQ[a^2 - b^2, 0] && IntegersQ[m, n] && GeQ[n - m, 0] && GtQ[m\*n, 0]

#### Rule 2607

Int[sec[(e\_.) + (f\_.)\*(x\_.)]^m\_.\*((b\_.)\*tan[(e\_.) + (f\_.)\*(x\_.)]^n\_.), x\_Symbol] :> Dist[1/f, Subst[Int[(b\*x)^n\*(1 + x^2)^(m/2 - 1), x], x, Tan[e + f\*x]], x] /; FreeQ[{b, e, f, n}, x] && IntegerQ[m/2] && !(IntegerQ[(n - 1)/2] && LtQ[0, n, m - 1])

#### Rule 30

Int[(x\_)^(m\_.), x\_Symbol] := Simp[x^(m + 1)/(m + 1), x] /; FreeQ[m, x] && NeQ[m, -1]

### Rubi steps

$$\begin{aligned} \int \csc^2(c + dx)(a + a \csc(c + dx))(A - A \csc(c + dx)) dx &= - \left( (aA) \int \cot^2(c + dx) \csc^2(c + dx) dx \right) \\ &= - \frac{(aA) \text{Subst} \left( \int x^2 dx, x, -\cot(c + dx) \right)}{d} \\ &= \frac{aA \cot^3(c + dx)}{3d} \end{aligned}$$

**Mathematica [A]** time = 0.0121646, size = 17, normalized size = 1.

$$\frac{aA \cot^3(c + dx)}{3d}$$

Antiderivative was successfully verified.

[In] Integrate[Csc[c + d\*x]^2\*(a + a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x]), x]

[Out] (a\*A\*Cot[c + d\*x]^3)/(3\*d)

**Maple [B]** time = 0.021, size = 38, normalized size = 2.2

$$\frac{1}{d} \left( -Aa \cot(dx + c) - Aa \left( -\frac{2}{3} - \frac{(\csc(dx + c))^2}{3} \right) \cot(dx + c) \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int(csc(d\*x+c)^2\*(a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)), x)

[Out] 1/d\*(-A\*a\*cot(d\*x+c)-A\*a\*(-2/3-1/3\*csc(d\*x+c)^2)\*cot(d\*x+c))

**Maxima [B]** time = 1.02169, size = 57, normalized size = 3.35

$$-\frac{\frac{3Aa}{\tan(dx+c)} - \frac{(3 \tan(dx+c)^2 + 1)Aa}{\tan(dx+c)^3}}{3d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^2\*(a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x, algorithm="maxima")

[Out]  $-1/3*(3*A*a/\tan(dx + c) - (3*\tan(dx + c)^2 + 1)*A*a/\tan(dx + c)^3)/d$

**Fricas [B]** time = 0.45574, size = 85, normalized size = 5.

$$\frac{Aa \cos(dx + c)^3}{3(d \cos(dx + c)^2 - d) \sin(dx + c)}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^2\*(a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x, algorithm="fricas")

[Out]  $-1/3*A*a*\cos(dx + c)^3/((d*\cos(dx + c)^2 - d)*\sin(dx + c))$

**Sympy [A]** time = 2.14552, size = 54, normalized size = 3.18

$$\begin{cases} \frac{Aa\left(-\frac{\cot^3(c+dx)}{3}-\cot(c+dx)\right)+Aa\cot(c+dx)}{d} & \text{for } d \neq 0 \\ x(-A\csc(c)+A)(a\csc(c)+a)\csc^2(c) & \text{otherwise} \end{cases}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*\*2\*(a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x)

[Out] Piecewise((-A\*a\*(-cot(c + d\*x)\*\*3/3 - cot(c + d\*x)) + A\*a\*cot(c + d\*x))/d, Ne(d, 0)), (x\*(-A\*csc(c) + A)\*(a\*csc(c) + a)\*csc(c)\*\*2, True))

**Giac [A]** time = 1.38489, size = 20, normalized size = 1.18

$$\frac{Aa}{3d \tan(dx + c)^3}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^2*(a+a*csc(d*x+c))*(A-A*csc(d*x+c)),x, algorithm="giac")
```

```
[Out] 1/3*A*a/(d*tan(d*x + c)^3)
```

### 3.15 $\int \csc(c+dx)(a+a \csc(c+dx))(A-A \csc(c+dx)) dx$

**Optimal.** Leaf size=38

$$\frac{aA \cot(c+dx) \csc(c+dx)}{2d} - \frac{aA \tanh^{-1}(\cos(c+dx))}{2d}$$

[Out]  $-(a*A*ArcTanh[Cos[c + d*x]])/(2*d) + (a*A*Cot[c + d*x]*Csc[c + d*x])/(2*d)$

**Rubi [A]** time = 0.0494713, antiderivative size = 38, normalized size of antiderivative = 1., number of steps used = 3, number of rules used = 3, integrand size = 28,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.107$ , Rules used = {3958, 2611, 3770}

$$\frac{aA \cot(c+dx) \csc(c+dx)}{2d} - \frac{aA \tanh^{-1}(\cos(c+dx))}{2d}$$

Antiderivative was successfully verified.

[In]  $\text{Int}[Csc[c + d*x]*(a + a*Csc[c + d*x])*(A - A*Csc[c + d*x]),x]$

[Out]  $-(a*A*ArcTanh[Cos[c + d*x]])/(2*d) + (a*A*Cot[c + d*x]*Csc[c + d*x])/(2*d)$

#### Rule 3958

$\text{Int}[\csc[(e_.) + (f_.)*(x_.)]*(\csc[(e_.) + (f_.)*(x_.)]*(b_.) + (a_.))^{(m_.)}*(\csc[(e_.) + (f_.)*(x_.)]*(d_.) + (c_.))^{(n_.)}, x\_Symbol] \rightarrow \text{Dist}[(-a*c)^m, \text{Int}[\text{ExpandTrig}[\csc[e + f*x]*\cot[e + f*x]^{(2*m)}, (c + d*\csc[e + f*x])^{(n - m)}, x], x] /; \text{FreeQ}\{a, b, c, d, e, f, n\}, x] \&\& \text{EqQ}[b*c + a*d, 0] \&\& \text{EqQ}[a^2 - b^2, 0] \&\& \text{IntegersQ}[m, n] \&\& \text{GeQ}[n - m, 0] \&\& \text{GtQ}[m*n, 0]$

#### Rule 2611

$\text{Int}[(a_.)*\sec[(e_.) + (f_.)*(x_.)]^{(m_.)}*((b_.)*\tan[(e_.) + (f_.)*(x_.)]^{(n_.)}, x\_Symbol] \rightarrow \text{Simp}[(b*(a*\sec[e + f*x])^m*(b*\tan[e + f*x])^{(n - 1)})/(f*(m + n - 1)), x] - \text{Dist}[(b^2*(n - 1))/(m + n - 1), \text{Int}[(a*\sec[e + f*x])^m*(b*\tan[e + f*x])^{(n - 2)}, x], x] /; \text{FreeQ}\{a, b, e, f, m\}, x] \&\& \text{GtQ}[n, 1] \&\& \text{NeQ}[m + n - 1, 0] \&\& \text{IntegersQ}[2*m, 2*n]$

#### Rule 3770

$\text{Int}[\csc[(c_.) + (d_.)*(x_.)], x\_Symbol] \rightarrow -\text{Simp}[\text{ArcTanh}[\text{Cos}[c + d*x]]/d, x] /; \text{FreeQ}\{c, d\}, x]$



Rubi steps

$$\begin{aligned} \int \csc(c + dx)(a + a \csc(c + dx))(A - A \csc(c + dx)) dx &= -\left( (aA) \int \cot^2(c + dx) \csc(c + dx) dx \right) \\ &= \frac{aA \cot(c + dx) \csc(c + dx)}{2d} + \frac{1}{2}(aA) \int \csc(c + dx) dx \\ &= -\frac{aA \tanh^{-1}(\cos(c + dx))}{2d} + \frac{aA \cot(c + dx) \csc(c + dx)}{2d} \end{aligned}$$

**Mathematica [B]** time = 0.0349487, size = 79, normalized size = 2.08

$$-aA \left( -\frac{\csc^2\left(\frac{1}{2}(c + dx)\right)}{8d} + \frac{\sec^2\left(\frac{1}{2}(c + dx)\right)}{8d} - \frac{\log\left(\sin\left(\frac{1}{2}(c + dx)\right)\right)}{2d} + \frac{\log\left(\cos\left(\frac{1}{2}(c + dx)\right)\right)}{2d} \right)$$

Antiderivative was successfully verified.

[In] Integrate[Csc[c + d\*x]\*(a + a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x]),x]

[Out] -(a\*A\*(-Csc[(c + d\*x)/2]^2/(8\*d) + Log[Cos[(c + d\*x)/2]]/(2\*d) - Log[Sin[(c + d\*x)/2]]/(2\*d) + Sec[(c + d\*x)/2]^2/(8\*d)))

**Maple [A]** time = 0.025, size = 44, normalized size = 1.2

$$\frac{Aa \ln(\csc(dx + c) - \cot(dx + c))}{2d} + \frac{Aa \cot(dx + c) \csc(dx + c)}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int(csc(d\*x+c)\*(a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x)

[Out] 1/2/d\*A\*a\*ln(csc(d\*x+c)-cot(d\*x+c))+1/2\*a\*A\*cot(d\*x+c)\*csc(d\*x+c)/d

**Maxima [A]** time = 1.00066, size = 92, normalized size = 2.42

$$\frac{Aa \left( \frac{2 \cos(dx+c)}{\cos(dx+c)^2-1} - \log(\cos(dx+c)+1) + \log(\cos(dx+c)-1) \right) + 4 Aa \log(\cot(dx+c) + \csc(dx+c))}{4d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x, algorithm="maxima")

[Out]  $-1/4*(A*a*(2*\cos(d*x + c)/(\cos(d*x + c)^2 - 1) - \log(\cos(d*x + c) + 1) + \log(\cos(d*x + c) - 1)) + 4*A*a*\log(\cot(d*x + c) + \csc(d*x + c)))/d$

**Fricas [B]** time = 0.495563, size = 223, normalized size = 5.87

$$\frac{2 A a \cos(dx + c) + (A a \cos(dx + c)^2 - A a) \log\left(\frac{1}{2} \cos(dx + c) + \frac{1}{2}\right) - (A a \cos(dx + c)^2 - A a) \log\left(-\frac{1}{2} \cos(dx + c) + \frac{1}{2}\right)}{4(d \cos(dx + c)^2 - d)}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x, algorithm="fricas")

[Out]  $-1/4*(2*A*a*\cos(d*x + c) + (A*a*\cos(d*x + c)^2 - A*a)*\log(1/2*\cos(d*x + c) + 1/2) - (A*a*\cos(d*x + c)^2 - A*a)*\log(-1/2*\cos(d*x + c) + 1/2))/(d*\cos(d*x + c)^2 - d)$

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$-Aa \left( \int -\csc(c + dx) dx + \int \csc^3(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x)

[Out]  $-A*a*(\text{Integral}(-\csc(c + d*x), x) + \text{Integral}(\csc(c + d*x)**3, x))$

**Giac [B]** time = 1.41278, size = 136, normalized size = 3.58

$$\frac{2 A a \log\left(\frac{|-\cos(dx+c)+1|}{|\cos(dx+c)+1|}\right) + \frac{A a (\cos(dx+c)-1)}{\cos(dx+c)+1} - \frac{\left(A a + \frac{2 A a (\cos(dx+c)-1)}{\cos(dx+c)+1}\right) (\cos(dx+c)+1)}{\cos(dx+c)-1}}{8 d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)*(a+a*csc(d*x+c))*(A-A*csc(d*x+c)),x, algorithm="giac")
```

```
[Out] 1/8*(2*A*a*log(abs(-cos(d*x + c) + 1)/abs(cos(d*x + c) + 1)) + A*a*(cos(d*x + c) - 1)/(cos(d*x + c) + 1) - (A*a + 2*A*a*(cos(d*x + c) - 1)/(cos(d*x + c) + 1))*(cos(d*x + c) + 1)/(cos(d*x + c) - 1))/d
```

### 3.16 $\int (a + a \csc(c + dx))(A - A \csc(c + dx)) \sin(c + dx) dx$

**Optimal.** Leaf size=27

$$\frac{aA \tanh^{-1}(\cos(c + dx))}{d} - \frac{aA \cos(c + dx)}{d}$$

[Out] (a\*A\*ArcTanh[Cos[c + d\*x]])/d - (a\*A\*Cos[c + d\*x])/d

**Rubi [A]** time = 0.0522252, antiderivative size = 27, normalized size of antiderivative = 1., number of steps used = 4, number of rules used = 4, integrand size = 28,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$ , Rules used = {3962, 2592, 321, 206}

$$\frac{aA \tanh^{-1}(\cos(c + dx))}{d} - \frac{aA \cos(c + dx)}{d}$$

Antiderivative was successfully verified.

[In] Int[(a + a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x])\*Sin[c + d\*x],x]

[Out] (a\*A\*ArcTanh[Cos[c + d\*x]])/d - (a\*A\*Cos[c + d\*x])/d

#### Rule 3962

Int[(csc[(e\_.) + (f\_.)\*(x\_.)]\*(g\_.))^(p\_.)\*(csc[(e\_.) + (f\_.)\*(x\_.)]\*(b\_.) + (a\_.)^(m\_.)\*(csc[(e\_.) + (f\_.)\*(x\_.)]\*(d\_.) + (c\_.))^(n\_.), x\_Symbol] :> Dist[(- (a\*c))^m, Int[ExpandTrig[(g\*csc[e + f\*x])^p\*cot[e + f\*x]^(2\*m), (c + d\*csc[e + f\*x])^(n - m), x], x], x] /; FreeQ[{a, b, c, d, e, f, g, n, p}, x] && EqQ[b\*c + a\*d, 0] && EqQ[a^2 - b^2, 0] && IntegersQ[m, n] && GeQ[n - m, 0] && GtQ[m\*n, 0]

#### Rule 2592

Int[((a\_.)\*sin[(e\_.) + (f\_.)\*(x\_.)])^(m\_.)\*tan[(e\_.) + (f\_.)\*(x\_.)]^(n\_.), x\_Symbol] :> With[{ff = FreeFactors[Sin[e + f\*x], x]}, Dist[ff/f, Subst[Int[(ff\*x)^(m + n)/(a^2 - ff^2\*x^2)^((n + 1)/2), x], x, (a\*Sin[e + f\*x])/ff], x] /; FreeQ[{a, e, f, m}, x] && IntegerQ[(n + 1)/2]

#### Rule 321

Int[((c\_.)\*(x\_.))^(m\_.)\*((a\_.) + (b\_.)\*(x\_.)^(n\_.))^(p\_.), x\_Symbol] :> Simp[(c^(n - 1)\*(c\*x)^(m - n + 1)\*(a + b\*x^n)^(p + 1))/(b\*(m + n\*p + 1)), x] - Dist[

```
(a*c^n*(m - n + 1))/(b*(m + n*p + 1)), Int[(c*x)^(m - n)*(a + b*x^n)^p, x],
x] /; FreeQ[{a, b, c, p}, x] && IGtQ[n, 0] && GtQ[m, n - 1] && NeQ[m + n*p
+ 1, 0] && IntBinomialQ[a, b, c, n, m, p, x]
```

### Rule 206

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] := Simp[(1*ArcTanh[(Rt[-b, 2]*x)/
Rt[a, 2]])/(Rt[a, 2]*Rt[-b, 2]), x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

### Rubi steps

$$\begin{aligned} \int (a + a \csc(c + dx))(A - A \csc(c + dx)) \sin(c + dx) dx &= -((aA) \int \cos(c + dx) \cot(c + dx) dx) \\ &= \frac{(aA) \operatorname{Subst}\left(\int \frac{x^2}{1-x^2} dx, x, \cos(c + dx)\right)}{d} \\ &= -\frac{aA \cos(c + dx)}{d} + \frac{(aA) \operatorname{Subst}\left(\int \frac{1}{1-x^2} dx, x, \cos(c + dx)\right)}{d} \\ &= \frac{aA \tanh^{-1}(\cos(c + dx))}{d} - \frac{aA \cos(c + dx)}{d} \end{aligned}$$

**Mathematica [A]** time = 0.022104, size = 46, normalized size = 1.7

$$-aA \left( \frac{\cos(c + dx)}{d} + \frac{\log\left(\sin\left(\frac{1}{2}(c + dx)\right)\right)}{d} - \frac{\log\left(\cos\left(\frac{1}{2}(c + dx)\right)\right)}{d} \right)$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + a*Csc[c + d*x])*(A - A*Csc[c + d*x])*Sin[c + d*x], x]
```

```
[Out] -(a*A*(Cos[c + d*x]/d - Log[Cos[(c + d*x)/2]]/d + Log[Sin[(c + d*x)/2]]/d))
```

**Maple [A]** time = 0.023, size = 38, normalized size = 1.4

$$-\frac{Aa \cos(dx + c)}{d} - \frac{Aa \ln(\csc(dx + c) - \cot(dx + c))}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `int((a+a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c),x)`

[Out] `-a*A*cos(d*x+c)/d-1/d*A*a*ln(csc(d*x+c)-cot(d*x+c))`

**Maxima [A]** time = 1.01463, size = 45, normalized size = 1.67

$$-\frac{Aa \cos(dx + c) - Aa \log(\cot(dx + c) + \csc(dx + c))}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a+a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c),x, algorithm="maxima")`

[Out] `-(A*a*cos(d*x + c) - A*a*log(cot(d*x + c) + csc(d*x + c)))/d`

**Fricas [A]** time = 0.506677, size = 132, normalized size = 4.89

$$\frac{2Aa \cos(dx + c) - Aa \log\left(\frac{1}{2} \cos(dx + c) + \frac{1}{2}\right) + Aa \log\left(-\frac{1}{2} \cos(dx + c) + \frac{1}{2}\right)}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a+a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c),x, algorithm="fricas")`

[Out] `-1/2*(2*A*a*cos(d*x + c) - A*a*log(1/2*cos(d*x + c) + 1/2) + A*a*log(-1/2*cos(d*x + c) + 1/2))/d`

**Sympy [A]** time = 11.4959, size = 75, normalized size = 2.78

$$Aa \left( \begin{cases} -\frac{\cot(c+dx)}{d \csc(c+dx)} & \text{for } d \neq 0 \\ \frac{x}{\csc(c)} & \text{otherwise} \end{cases} \right) - Aa \left( \begin{cases} \frac{x \cot(c) \csc(c)}{\cot(c)+\csc(c)} + \frac{x \csc^2(c)}{\cot(c)+\csc(c)} & \text{for } d = 0 \\ -\frac{\log(\cot(c+dx)+\csc(c+dx))}{d} & \text{otherwise} \end{cases} \right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c),x)
```

```
[Out] A*a*Piecewise((-cot(c + d*x)/(d*csc(c + d*x)), Ne(d, 0)), (x/csc(c), True))
- A*a*Piecewise((x*cot(c)*csc(c)/(cot(c) + csc(c)) + x*csc(c)**2/(cot(c) +
csc(c)), Eq(d, 0)), (-log(cot(c + d*x) + csc(c + d*x))/d, True))
```

**Giac [B]** time = 1.39327, size = 81, normalized size = 3.

$$-\frac{Aa \log\left(\frac{|-\cos(dx+c)+1|}{|\cos(dx+c)+1|}\right) - \frac{4Aa}{\frac{\cos(dx+c)-1}{\cos(dx+c)+1}-1}}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c),x, algorithm="giac")
```

```
[Out] -1/2*(A*a*log(abs(-cos(d*x + c) + 1)/abs(cos(d*x + c) + 1)) - 4*A*a/((cos(d
*x + c) - 1)/(cos(d*x + c) + 1) - 1))/d
```

### 3.17 $\int (a + a \csc(c + dx))(A - A \csc(c + dx)) \sin^2(c + dx) dx$

**Optimal.** Leaf size=29

$$-\frac{aA \sin(c + dx) \cos(c + dx)}{2d} - \frac{1}{2}aAx$$

[Out]  $-(aAx)/2 - (aA \cos[c + dx] \sin[c + dx])/(2d)$

**Rubi [A]** time = 0.0551765, antiderivative size = 29, normalized size of antiderivative = 1., number of steps used = 3, number of rules used = 3, integrand size = 30,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.1$ , Rules used = {3962, 2635, 8}

$$-\frac{aA \sin(c + dx) \cos(c + dx)}{2d} - \frac{1}{2}aAx$$

Antiderivative was successfully verified.

[In]  $\text{Int}[(a + a \csc[c + dx])*(A - A \csc[c + dx])* \sin[c + dx]^2, x]$

[Out]  $-(aAx)/2 - (aA \cos[c + dx] \sin[c + dx])/(2d)$

#### Rule 3962

```
Int[(csc[(e_.) + (f_.)*(x_.)]*(g_.))^(p_.)*(csc[(e_.) + (f_.)*(x_.)]*(b_.) +
(a_.)^(m_.)*(csc[(e_.) + (f_.)*(x_.)]*(d_.) + (c_.)^(n_.), x_Symbol] := Dist
[(-a*c)^(m), Int[ExpandTrig[(g*csc[e + f*x])^p*cot[e + f*x]^(2*m), (c + d*c
sc[e + f*x])^(n - m), x], x], x] /; FreeQ[{a, b, c, d, e, f, g, n, p}, x] &
& EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && IntegersQ[m, n] && GeQ[n - m, 0
] && GtQ[m*n, 0]
```

#### Rule 2635

```
Int[((b_.)*sin[(c_.) + (d_.)*(x_.)])^(n_.), x_Symbol] := -Simp[(b*cos[c + d*x]
)*(b*sin[c + d*x])^(n - 1)/(d*n), x] + Dist[(b^2*(n - 1))/n, Int[(b*sin[c
+ d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n
]
```

#### Rule 8

```
Int[a_, x_Symbol] := Simp[a*x, x] /; FreeQ[a, x]
```



Rubi steps

$$\begin{aligned} \int (a + a \csc(c + dx))(A - A \csc(c + dx)) \sin^2(c + dx) dx &= - \left( (aA) \int \cos^2(c + dx) dx \right) \\ &= - \frac{aA \cos(c + dx) \sin(c + dx)}{2d} - \frac{1}{2}(aA) \int 1 dx \\ &= -\frac{1}{2}aAx - \frac{aA \cos(c + dx) \sin(c + dx)}{2d} \end{aligned}$$

**Mathematica [A]** time = 0.0113862, size = 25, normalized size = 0.86

$$\frac{aA(2(c + dx) + \sin(2(c + dx)))}{4d}$$

Antiderivative was successfully verified.

[In] Integrate[(a + a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x])\*Sin[c + d\*x]^2,x]

[Out] -(a\*A\*(2\*(c + d\*x) + Sin[2\*(c + d\*x)]))/(4\*d)

**Maple [A]** time = 0.021, size = 40, normalized size = 1.4

$$\frac{1}{d} \left( Aa \left( -\frac{\cos(dx + c) \sin(dx + c)}{2} + \frac{dx}{2} + \frac{c}{2} \right) - Aa(dx + c) \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int((a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c))/csc(d\*x+c)^2,x)

[Out] 1/d\*(A\*a\*(-1/2\*cos(d\*x+c)\*sin(d\*x+c)+1/2\*d\*x+1/2\*c)-A\*a\*(d\*x+c))

**Maxima [A]** time = 0.993512, size = 50, normalized size = 1.72

$$\frac{(2 dx + 2 c - \sin(2 dx + 2 c))Aa - 4 (dx + c)Aa}{4 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c))/csc(d\*x+c)^2,x, algorithm="maxima")

[Out] 1/4\*((2\*d\*x + 2\*c - sin(2\*d\*x + 2\*c))\*A\*a - 4\*(d\*x + c)\*A\*a)/d

**Fricas [A]** time = 0.479009, size = 68, normalized size = 2.34

$$\frac{Aadx + Aa \cos(dx + c) \sin(dx + c)}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c))/csc(d\*x+c)^2,x, algorithm="fricas")

[Out] -1/2\*(A\*a\*d\*x + A\*a\*cos(d\*x + c)\*sin(d\*x + c))/d

**Sympy [A]** time = 6.96551, size = 85, normalized size = 2.93

$$\begin{cases} \frac{Aax \cot^2(c+dx)}{2 \csc^2(c+dx)} - Aax + \frac{Aax}{2 \csc^2(c+dx)} - \frac{Aa \cot(c+dx)}{2d \csc^2(c+dx)} & \text{for } d \neq 0 \\ \frac{x(-A \csc(c)+A)(a \csc(c)+a)}{\csc^2(c)} & \text{otherwise} \end{cases}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c))/csc(d\*x+c)\*\*2,x)

[Out] Piecewise((A\*a\*x\*cot(c + d\*x)\*\*2/(2\*csc(c + d\*x)\*\*2) - A\*a\*x + A\*a\*x/(2\*csc(c + d\*x)\*\*2) - A\*a\*cot(c + d\*x)/(2\*d\*csc(c + d\*x)\*\*2), Ne(d, 0)), (x\*(-A\*csc(c) + A)\*(a\*csc(c) + a)/csc(c)\*\*2, True))

**Giac [A]** time = 1.45297, size = 47, normalized size = 1.62

$$\frac{(dx + c)Aa + \frac{Aa \tan(dx+c)}{\tan(dx+c)^2+1}}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c)^2,x, algorithm="giac")
```

```
[Out] -1/2*((d*x + c)*A*a + A*a*tan(d*x + c)/(tan(d*x + c)^2 + 1))/d
```

### 3.18 $\int (a + a \csc(c + dx))(A - A \csc(c + dx)) \sin^3(c + dx) dx$

Optimal. Leaf size=17

$$\frac{aA \cos^3(c + dx)}{3d}$$

[Out] (a\*A\*Cos[c + d\*x]^3)/(3\*d)

**Rubi [A]** time = 0.0606128, antiderivative size = 17, normalized size of antiderivative = 1., number of steps used = 3, number of rules used = 3, integrand size = 30,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.1$ , Rules used = {3962, 2565, 30}

$$\frac{aA \cos^3(c + dx)}{3d}$$

Antiderivative was successfully verified.

[In] Int[(a + a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x])\*Sin[c + d\*x]^3,x]

[Out] (a\*A\*Cos[c + d\*x]^3)/(3\*d)

#### Rule 3962

```
Int[(csc[(e_.) + (f_.)*(x_.)]*(g_.))^(p_.)*(csc[(e_.) + (f_.)*(x_.)]*(b_.) +
(a_.)^(m_.)*(csc[(e_.) + (f_.)*(x_.)]*(d_.) + (c_.))^(n_.), x_Symbol] :> Dist
[(-a*c)^(m), Int[ExpandTrig[(g*csc[e + f*x])^p*cot[e + f*x]^(2*m), (c + d*c
sc[e + f*x])^(n - m), x], x] /; FreeQ[{a, b, c, d, e, f, g, n, p}, x] &
& EqQ[b*c + a*d, 0] && EqQ[a^2 - b^2, 0] && IntegersQ[m, n] && GeQ[n - m, 0
] && GtQ[m*n, 0]
```

#### Rule 2565

```
Int[(cos[(e_.) + (f_.)*(x_.)]*(a_.))^(m_.)*sin[(e_.) + (f_.)*(x_.)]^(n_.), x_
Symbol] :> -Dist[(a*f)^(-1), Subst[Int[x^m*(1 - x^2/a^2)^((n - 1)/2), x], x
, a*Cos[e + f*x]], x] /; FreeQ[{a, e, f, m}, x] && IntegerQ[(n - 1)/2] &&
!(IntegerQ[(m - 1)/2] && GtQ[m, 0] && LeQ[m, n])
```

#### Rule 30

```
Int[(x_)^(m_.), x_Symbol] :> Simp[x^(m + 1)/(m + 1), x] /; FreeQ[m, x] && N
eQ[m, -1]
```

Rubi steps

$$\begin{aligned} \int (a + a \csc(c + dx))(A - A \csc(c + dx)) \sin^3(c + dx) dx &= -\left((aA) \int \cos^2(c + dx) \sin(c + dx) dx\right) \\ &= \frac{(aA) \operatorname{Subst}\left(\int x^2 dx, x, \cos(c + dx)\right)}{d} \\ &= \frac{aA \cos^3(c + dx)}{3d} \end{aligned}$$

**Mathematica [A]** time = 0.0056628, size = 17, normalized size = 1.

$$\frac{aA \cos^3(c + dx)}{3d}$$

Antiderivative was successfully verified.

[In] Integrate[(a + a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x])\*Sin[c + d\*x]^3,x]

[Out] (a\*A\*Cos[c + d\*x]^3)/(3\*d)

**Maple [B]** time = 0.028, size = 35, normalized size = 2.1

$$\frac{1}{d} \left( -\frac{Aa(2 + (\sin(dx + c))^2) \cos(dx + c)}{3} + Aa \cos(dx + c) \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int((a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c))/csc(d\*x+c)^3,x)

[Out] 1/d\*(-1/3\*A\*a\*(2+sin(d\*x+c)^2)\*cos(d\*x+c)+A\*a\*cos(d\*x+c))

**Maxima [B]** time = 0.981429, size = 49, normalized size = 2.88

$$\frac{(\cos(dx + c)^3 - 3 \cos(dx + c))Aa + 3Aa \cos(dx + c)}{3d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c))/csc(d\*x+c)^3,x, algorithm="maxima")

[Out] 1/3\*((cos(d\*x + c)^3 - 3\*cos(d\*x + c))\*A\*a + 3\*A\*a\*cos(d\*x + c))/d

**Fricas [A]** time = 0.472044, size = 35, normalized size = 2.06

$$\frac{Aa \cos(dx + c)^3}{3d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c))/csc(d\*x+c)^3,x, algorithm="fricas")

[Out] 1/3\*A\*a\*cos(d\*x + c)^3/d

**Sympy [A]** time = 10.5602, size = 83, normalized size = 4.88

$$\begin{cases} -\frac{2Aa \cot^3(c+dx)}{3d \csc^3(c+dx)} + \frac{Aa \cot(c+dx)}{d \csc(c+dx)} - \frac{Aa \cot(c+dx)}{d \csc^3(c+dx)} & \text{for } d \neq 0 \\ \frac{x(-A \csc(c)+A)(a \csc(c)+a)}{\csc^3(c)} & \text{otherwise} \end{cases}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a+a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c))/csc(d\*x+c)\*\*3,x)

[Out] Piecewise((-2\*A\*a\*cot(c + d\*x)\*\*3/(3\*d\*csc(c + d\*x)\*\*3) + A\*a\*cot(c + d\*x)/(d\*csc(c + d\*x)) - A\*a\*cot(c + d\*x)/(d\*csc(c + d\*x)\*\*3), Ne(d, 0)), (x\*(-A\*csc(c) + A)\*(a\*csc(c) + a)/csc(c)\*\*3, True))

**Giac [B]** time = 1.31462, size = 76, normalized size = 4.47

$$\frac{2 \left( Aa + \frac{3Aa(\cos(dx+c)-1)^2}{(\cos(dx+c)+1)^2} \right)}{3d \left( \frac{\cos(dx+c)-1}{\cos(dx+c)+1} - 1 \right)^3}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a+a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c)^3,x, algorithm="giac")
```

```
[Out] -2/3*(A*a + 3*A*a*(cos(d*x + c) - 1)^2/(cos(d*x + c) + 1)^2)/(d*((cos(d*x + c) - 1)/(cos(d*x + c) + 1) - 1)^3)
```

### 3.19 $\int \csc^3(c+dx)(a-a \csc(c+dx))(A-A \csc(c+dx)) dx$

**Optimal.** Leaf size=91

$$\frac{2aA \cot^3(c+dx)}{3d} + \frac{2aA \cot(c+dx)}{d} - \frac{7aA \tanh^{-1}(\cos(c+dx))}{8d} - \frac{aA \cot(c+dx) \csc^3(c+dx)}{4d} - \frac{7aA \cot(c+dx) \csc(c+dx)}{8d}$$

[Out]  $(-7*a*A*ArcTanh[Cos[c + d*x]])/(8*d) + (2*a*A*Cot[c + d*x])/d + (2*a*A*Cot[c + d*x]^3)/(3*d) - (7*a*A*Cot[c + d*x]*Csc[c + d*x])/(8*d) - (a*A*Cot[c + d*x]*Csc[c + d*x]^3)/(4*d)$

**Rubi [A]** time = 0.0959867, antiderivative size = 91, normalized size of antiderivative = 1., number of steps used = 7, number of rules used = 6, integrand size = 31,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.194$ , Rules used = {21, 3788, 3767, 4046, 3768, 3770}

$$\frac{2aA \cot^3(c+dx)}{3d} + \frac{2aA \cot(c+dx)}{d} - \frac{7aA \tanh^{-1}(\cos(c+dx))}{8d} - \frac{aA \cot(c+dx) \csc^3(c+dx)}{4d} - \frac{7aA \cot(c+dx) \csc(c+dx)}{8d}$$

Antiderivative was successfully verified.

[In]  $\text{Int}[Csc[c + d*x]^3*(a - a*Csc[c + d*x])*(A - A*Csc[c + d*x]),x]$

[Out]  $(-7*a*A*ArcTanh[Cos[c + d*x]])/(8*d) + (2*a*A*Cot[c + d*x])/d + (2*a*A*Cot[c + d*x]^3)/(3*d) - (7*a*A*Cot[c + d*x]*Csc[c + d*x])/(8*d) - (a*A*Cot[c + d*x]*Csc[c + d*x]^3)/(4*d)$

#### Rule 21

$\text{Int}[(u_.)*((a_.) + (b_.)*(v_.))^(m_.)*((c_.) + (d_.)*(v_.))^(n_.), x\_Symbol] :>$   
 $\text{Dist}[(b/d)^m, \text{Int}[u*(c + d*v)^(m + n), x], x] /;$  FreeQ[{a, b, c, d, n}, x]  
 && EqQ[b\*c - a\*d, 0] && IntegerQ[m] && (!IntegerQ[n] || SimplerQ[c + d\*x, a + b\*x])

#### Rule 3788

$\text{Int}[(\csc[(e_.) + (f_.)*(x_.)]*(d_.))^(n_.)*(\csc[(e_.) + (f_.)*(x_.)]*(b_.) + (a_.))^2, x\_Symbol] :>$  Dist[(2\*a\*b)/d, Int[(d\*Csc[e + f\*x])^(n + 1), x], x]  
 + Int[(d\*Csc[e + f\*x])^n\*(a^2 + b^2\*Csc[e + f\*x]^2), x] /; FreeQ[{a, b, d, e, f, n}, x]

#### Rule 3767



```
Int[csc[(c_.) + (d_.)*(x_)]^(n_), x_Symbol] := -Dist[d^(-1), Subst[Int[ExpandIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d*x]], x] /; FreeQ[{c, d}, x] && IGtQ[n/2, 0]
```

### Rule 4046

```
Int[(csc[(e_.) + (f_.)*(x_)]*(b_.))^(m_.)*(csc[(e_.) + (f_.)*(x_)]^2*(C_.) + (A_.)), x_Symbol] := -Simp[(C*Cot[e + f*x]*(b*Csc[e + f*x])^m)/(f*(m + 1)), x] + Dist[(C*m + A*(m + 1))/(m + 1), Int[(b*Csc[e + f*x])^m, x], x] /; FreeQ[{b, e, f, A, C, m}, x] && NeQ[C*m + A*(m + 1), 0] && !LeQ[m, -1]
```

### Rule 3768

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_), x_Symbol] := -Simp[(b*Cos[c + d*x]*(b*Csc[c + d*x])^(n - 1))/(d*(n - 1)), x] + Dist[(b^2*(n - 2))/(n - 1), Int[(b*Csc[c + d*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2*n]
```

### Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_)], x_Symbol] := -Simp[ArcTanh[Cos[c + d*x]]/d, x] /; FreeQ[{c, d}, x]
```

### Rubi steps

$$\begin{aligned} \int \csc^3(c + dx)(a - a \csc(c + dx))(A - A \csc(c + dx)) dx &= \frac{A \int \csc^3(c + dx)(a - a \csc(c + dx))^2 dx}{a} \\ &= \frac{A \int \csc^3(c + dx)(a^2 + a^2 \csc^2(c + dx)) dx}{a} - (2aA) \int \csc^4(c + dx) dx \\ &= -\frac{aA \cot(c + dx) \csc^3(c + dx)}{4d} + \frac{1}{4}(7aA) \int \csc^3(c + dx) dx \\ &= \frac{2aA \cot(c + dx)}{d} + \frac{2aA \cot^3(c + dx)}{3d} - \frac{7aA \cot(c + dx) \csc(c + dx)}{8d} \\ &= -\frac{7aA \tanh^{-1}(\cos(c + dx))}{8d} + \frac{2aA \cot(c + dx)}{d} + \frac{2aA \cot^3(c + dx)}{3d} \end{aligned}$$

**Mathematica [A]** time = 0.0574015, size = 163, normalized size = 1.79

$$\frac{4aA \cot(c + dx)}{3d} - \frac{aA \csc^4\left(\frac{1}{2}(c + dx)\right)}{64d} - \frac{7aA \csc^2\left(\frac{1}{2}(c + dx)\right)}{32d} + \frac{aA \sec^4\left(\frac{1}{2}(c + dx)\right)}{64d} + \frac{7aA \sec^2\left(\frac{1}{2}(c + dx)\right)}{32d} + \frac{7aA}{32d}$$

Antiderivative was successfully verified.

[In] Integrate[Csc[c + d\*x]^3\*(a - a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x]),x]

[Out]  $(4*a*A*\cot[c + d*x])/(3*d) - (7*a*A*\csc[(c + d*x)/2]^2)/(32*d) - (a*A*\csc[(c + d*x)/2]^4)/(64*d) + (2*a*A*\cot[c + d*x]*\csc[c + d*x]^2)/(3*d) - (7*a*A*\log[\cos[(c + d*x)/2]])/(8*d) + (7*a*A*\log[\sin[(c + d*x)/2]])/(8*d) + (7*a*A*\sec[(c + d*x)/2]^2)/(32*d) + (a*A*\sec[(c + d*x)/2]^4)/(64*d)$

**Maple [A]** time = 0.049, size = 99, normalized size = 1.1

$$-\frac{7 A a \cot (d x+c) \csc (d x+c)}{8 d} + \frac{7 A a \ln (\csc (d x+c)-\cot (d x+c))}{8 d} + \frac{4 A a \cot (d x+c)}{3 d} + \frac{2 A a \cot (d x+c) (\csc (d x+c))}{3 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int(csc(d\*x+c)^3\*(a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x)

[Out]  $-7/8*a*A*\cot(d*x+c)*\csc(d*x+c)/d+7/8/d*A*a*\ln(\csc(d*x+c)-\cot(d*x+c))+4/3*a*A*\cot(d*x+c)/d+2/3*a*A*\cot(d*x+c)*\csc(d*x+c)^2/d-1/4*a*A*\cot(d*x+c)*\csc(d*x+c)^3/d$

**Maxima [A]** time = 0.988068, size = 196, normalized size = 2.15

$$\frac{3 A a \left( \frac{2 (3 \cos (d x+c)^3-5 \cos (d x+c))}{\cos (d x+c)^4-2 \cos (d x+c)^2+1} - 3 \log (\cos (d x+c)+1) + 3 \log (\cos (d x+c)-1) \right) + 12 A a \left( \frac{2 \cos (d x+c)}{\cos (d x+c)^2-1} - \log (\cos (d x+c)+1) \right)}{48 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^3\*(a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x, algorithm="maxima")

[Out]  $1/48*(3*A*a*(2*(3*\cos(d*x + c)^3 - 5*\cos(d*x + c))/(\cos(d*x + c)^4 - 2*\cos(d*x + c)^2 + 1) - 3*\log(\cos(d*x + c) + 1) + 3*\log(\cos(d*x + c) - 1)) + 12*A*a*(2*\cos(d*x + c)/(\cos(d*x + c)^2 - 1) - \log(\cos(d*x + c) + 1) + \log(\cos(d*x + c) - 1)) + 32*(3*\tan(d*x + c)^2 + 1)*A*a/\tan(d*x + c)^3)/d$

**Fricas [A]** time = 0.504306, size = 439, normalized size = 4.82

$$\frac{42 Aa \cos(dx + c)^3 - 54 Aa \cos(dx + c) - 21 \left( Aa \cos(dx + c)^4 - 2 Aa \cos(dx + c)^2 + Aa \right) \log\left(\frac{1}{2} \cos(dx + c) + \frac{1}{2}\right) + 21 \left( Aa \cos(dx + c)^4 - 2 Aa \cos(dx + c)^2 + Aa \right) \log\left(\frac{1}{2} \cos(dx + c) - \frac{1}{2}\right)}{48 \left( d \cos(dx + c)^2 + d \right)}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^3\*(a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x, algorithm="fricas")

[Out] 1/48\*(42\*A\*a\*cos(d\*x + c)^3 - 54\*A\*a\*cos(d\*x + c) - 21\*(A\*a\*cos(d\*x + c)^4 - 2\*A\*a\*cos(d\*x + c)^2 + A\*a)\*log(1/2\*cos(d\*x + c) + 1/2) + 21\*(A\*a\*cos(d\*x + c)^4 - 2\*A\*a\*cos(d\*x + c)^2 + A\*a)\*log(-1/2\*cos(d\*x + c) + 1/2) - 32\*(2\*A\*a\*cos(d\*x + c)^3 - 3\*A\*a\*cos(d\*x + c))\*sin(d\*x + c))/(d\*cos(d\*x + c)^4 - 2\*d\*cos(d\*x + c)^2 + d)

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$Aa \left( \int \csc^3(c + dx) dx + \int -2 \csc^4(c + dx) dx + \int \csc^5(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*\*3\*(a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x)

[Out] A\*a\*(Integral(csc(c + d\*x)\*\*3, x) + Integral(-2\*csc(c + d\*x)\*\*4, x) + Integral(csc(c + d\*x)\*\*5, x))

**Giac [A]** time = 1.47935, size = 209, normalized size = 2.3

$$3 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^4 - 16 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^3 + 48 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^2 + 168 Aa \log\left(\left|\tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)\right|\right) - 144 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)$$

192 d

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^3\*(a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x, algorithm="giac")

```
[Out] 1/192*(3*A*a*tan(1/2*d*x + 1/2*c)^4 - 16*A*a*tan(1/2*d*x + 1/2*c)^3 + 48*A*
a*tan(1/2*d*x + 1/2*c)^2 + 168*A*a*log(abs(tan(1/2*d*x + 1/2*c))) - 144*A*a
*tan(1/2*d*x + 1/2*c) - (350*A*a*tan(1/2*d*x + 1/2*c)^4 - 144*A*a*tan(1/2*d
*x + 1/2*c)^3 + 48*A*a*tan(1/2*d*x + 1/2*c)^2 - 16*A*a*tan(1/2*d*x + 1/2*c)
+ 3*A*a)/tan(1/2*d*x + 1/2*c)^4)/d
```

### 3.20 $\int \csc^2(c+dx)(a-a \csc(c+dx))(A-A \csc(c+dx)) dx$

**Optimal.** Leaf size=70

$$-\frac{5aA \cot(c+dx)}{3d} + \frac{aA \tanh^{-1}(\cos(c+dx))}{d} - \frac{aA \cot(c+dx) \csc^2(c+dx)}{3d} + \frac{aA \cot(c+dx) \csc(c+dx)}{d}$$

[Out] (a\*A\*ArcTanh[Cos[c + d\*x]])/d - (5\*a\*A\*Cot[c + d\*x])/(3\*d) + (a\*A\*Cot[c + d\*x]\*Csc[c + d\*x])/d - (a\*A\*Cot[c + d\*x]\*Csc[c + d\*x]^2)/(3\*d)

**Rubi [A]** time = 0.0898258, antiderivative size = 70, normalized size of antiderivative = 1., number of steps used = 7, number of rules used = 7, integrand size = 31,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.226$ , Rules used = {21, 3788, 3768, 3770, 4046, 3767, 8}

$$-\frac{5aA \cot(c+dx)}{3d} + \frac{aA \tanh^{-1}(\cos(c+dx))}{d} - \frac{aA \cot(c+dx) \csc^2(c+dx)}{3d} + \frac{aA \cot(c+dx) \csc(c+dx)}{d}$$

Antiderivative was successfully verified.

[In] Int[Csc[c + d\*x]^2\*(a - a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x]),x]

[Out] (a\*A\*ArcTanh[Cos[c + d\*x]])/d - (5\*a\*A\*Cot[c + d\*x])/(3\*d) + (a\*A\*Cot[c + d\*x]\*Csc[c + d\*x])/d - (a\*A\*Cot[c + d\*x]\*Csc[c + d\*x]^2)/(3\*d)

#### Rule 21

```
Int[(u_.)*((a_) + (b_.)*(v_.))^(m_.)*((c_) + (d_.)*(v_.))^(n_.), x_Symbol] :=
  Dist[(b/d)^m, Int[u*(c + d*v)^(m + n), x], x] /; FreeQ[{a, b, c, d, n}, x]
  && EqQ[b*c - a*d, 0] && IntegerQ[m] && (!IntegerQ[n] || SimplerQ[c + d*x,
  a + b*x])
```

#### Rule 3788

```
Int[(csc[(e_.) + (f_.)*(x_)]*(d_.))^(n_.)*(csc[(e_.) + (f_.)*(x_)]*(b_.) +
(a_.))^2, x_Symbol] := Dist[(2*a*b)/d, Int[(d*Csc[e + f*x])^(n + 1), x], x]
+ Int[(d*Csc[e + f*x])^n*(a^2 + b^2*Csc[e + f*x]^2), x] /; FreeQ[{a, b, d,
e, f, n}, x]
```

#### Rule 3768

```
Int[(csc[(c_.) + (d_.)*(x_)]*(b_.))^(n_.), x_Symbol] := -Simp[(b*Cos[c + d*x]
)*(b*Csc[c + d*x])^(n - 1)/(d*(n - 1)), x] + Dist[(b^2*(n - 2))/(n - 1), I
```

Int[(b\*Csc[c + d\*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2\*n]

### Rule 3770

Int[csc[(c\_.) + (d\_.)\*(x\_)], x\_Symbol] := -Simp[ArcTanh[Cos[c + d\*x]]/d, x] /; FreeQ[{c, d}, x]

### Rule 4046

Int[(csc[(e\_.) + (f\_.)\*(x\_)])\*(b\_.)^(m\_.)\*(csc[(e\_.) + (f\_.)\*(x\_)])^2\*(C\_. + (A\_.)), x\_Symbol] := -Simp[(C\*Cot[e + f\*x]\*(b\*Csc[e + f\*x])^m)/(f\*(m + 1)), x] + Dist[(C\*m + A\*(m + 1))/(m + 1), Int[(b\*Csc[e + f\*x])^m, x], x] /; FreeQ[{b, e, f, A, C, m}, x] && NeQ[C\*m + A\*(m + 1), 0] && !LeQ[m, -1]

### Rule 3767

Int[csc[(c\_.) + (d\_.)\*(x\_)^(n\_)], x\_Symbol] := -Dist[d^(-1), Subst[Int[ExpandIntegrand[(1 + x^2)^(n/2 - 1), x], x], x, Cot[c + d\*x]], x] /; FreeQ[{c, d}, x] && IGtQ[n/2, 0]

### Rule 8

Int[a\_, x\_Symbol] := Simp[a\*x, x] /; FreeQ[a, x]

### Rubi steps

$$\begin{aligned}
 \int \csc^2(c + dx)(a - a \csc(c + dx))(A - A \csc(c + dx)) dx &= \frac{A \int \csc^2(c + dx)(a - a \csc(c + dx))^2 dx}{a} \\
 &= \frac{A \int \csc^2(c + dx)(a^2 + a^2 \csc^2(c + dx)) dx}{a} - (2aA) \int \csc^3(c + dx) dx \\
 &= \frac{aA \cot(c + dx) \csc(c + dx)}{d} - \frac{aA \cot(c + dx) \csc^2(c + dx)}{3d} - \frac{aA \csc^3(c + dx)}{3d} \\
 &= \frac{aA \tanh^{-1}(\cos(c + dx))}{d} + \frac{aA \cot(c + dx) \csc(c + dx)}{d} - \frac{aA \csc^3(c + dx)}{3d} \\
 &= \frac{aA \tanh^{-1}(\cos(c + dx))}{d} - \frac{5aA \cot(c + dx)}{3d} + \frac{aA \cot(c + dx)}{d}
 \end{aligned}$$

**Mathematica [A]** time = 0.0426272, size = 116, normalized size = 1.66

$$-\frac{5aA \cot(c + dx)}{3d} + \frac{aA \csc^2\left(\frac{1}{2}(c + dx)\right)}{4d} - \frac{aA \sec^2\left(\frac{1}{2}(c + dx)\right)}{4d} - \frac{aA \log\left(\sin\left(\frac{1}{2}(c + dx)\right)\right)}{d} + \frac{aA \log\left(\cos\left(\frac{1}{2}(c + dx)\right)\right)}{d}$$

Antiderivative was successfully verified.

[In] Integrate[Csc[c + d\*x]^2\*(a - a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x]),x]

[Out]  $(-5*a*A*\cot[c + d*x])/(3*d) + (a*A*Csc[(c + d*x)/2]^2)/(4*d) - (a*A*\cot[c + d*x]*Csc[c + d*x]^2)/(3*d) + (a*A*\log[\cos[(c + d*x)/2]])/d - (a*A*\log[\sin[(c + d*x)/2]])/d - (a*A*\sec[(c + d*x)/2]^2)/(4*d)$

**Maple [A]** time = 0.045, size = 77, normalized size = 1.1

$$-\frac{5 A a \cot (d x+c)}{3 d} + \frac{A a \cot (d x+c) \csc (d x+c)}{d} - \frac{A a \ln (\csc (d x+c)-\cot (d x+c))}{d} - \frac{A a \cot (d x+c)(\csc (d x+c))^2}{3 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int(csc(d\*x+c)^2\*(a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x)

[Out]  $-5/3*a*A*\cot(d*x+c)/d+a*A*\cot(d*x+c)*\csc(d*x+c)/d-1/d*A*a*\ln(\csc(d*x+c)-\cot(d*x+c))-1/3*a*A*\cot(d*x+c)*\csc(d*x+c)^2/d$

**Maxima [A]** time = 1.05031, size = 117, normalized size = 1.67

$$\frac{3 A a \left( \frac{2 \cos (d x+c)}{\cos (d x+c)^2-1} - \log (\cos (d x+c)+1) + \log (\cos (d x+c)-1) \right) + \frac{6 A a}{\tan (d x+c)} + \frac{2(3 \tan (d x+c)^2+1) A a}{\tan (d x+c)^3}}{6 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)^2\*(a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x, algorithm="maxima")

[Out]  $-1/6*(3*A*a*(2*\cos(d*x + c)/(\cos(d*x + c)^2 - 1) - \log(\cos(d*x + c) + 1) + \log(\cos(d*x + c) - 1)) + 6*A*a/\tan(d*x + c) + 2*(3*\tan(d*x + c)^2 + 1)*A*a/\tan(d*x + c)^3)/d$

**Fricas [B]** time = 0.504544, size = 363, normalized size = 5.19

$$\frac{10 A a \cos (d x+c)^3 + 6 A a \cos (d x+c) \sin (d x+c) - 12 A a \cos (d x+c) - 3 \left( A a \cos (d x+c)^2 - A a \right) \log \left( \frac{1}{2} \cos (d x+c) + \frac{1}{2} \right)}{6 \left( d \cos (d x+c)^2 - d \right) \sin (d x+c)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^2*(a-a*csc(d*x+c))*(A-A*csc(d*x+c)),x, algorithm="fricas")
```

```
[Out] -1/6*(10*A*a*cos(d*x + c)^3 + 6*A*a*cos(d*x + c)*sin(d*x + c) - 12*A*a*cos(d*x + c) - 3*(A*a*cos(d*x + c)^2 - A*a)*log(1/2*cos(d*x + c) + 1/2)*sin(d*x + c) + 3*(A*a*cos(d*x + c)^2 - A*a)*log(-1/2*cos(d*x + c) + 1/2)*sin(d*x + c))/((d*cos(d*x + c)^2 - d)*sin(d*x + c))
```

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$Aa \left( \int \csc^2(c + dx) dx + \int -2 \csc^3(c + dx) dx + \int \csc^4(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)**2*(a-a*csc(d*x+c))*(A-A*csc(d*x+c)),x)
```

```
[Out] A*a*(Integral(csc(c + d*x)**2, x) + Integral(-2*csc(c + d*x)**3, x) + Integral(csc(c + d*x)**4, x))
```

**Giac [A]** time = 1.64671, size = 166, normalized size = 2.37

$$Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^3 - 6 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^2 - 24 Aa \log\left(\left|\tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)\right|\right) + 21 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right) + \frac{44 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)}{24 d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(csc(d*x+c)^2*(a-a*csc(d*x+c))*(A-A*csc(d*x+c)),x, algorithm="giac")
```

```
[Out] 1/24*(A*a*tan(1/2*d*x + 1/2*c)^3 - 6*A*a*tan(1/2*d*x + 1/2*c)^2 - 24*A*a*log(abs(tan(1/2*d*x + 1/2*c))) + 21*A*a*tan(1/2*d*x + 1/2*c) + (44*A*a*tan(1/2*d*x + 1/2*c)^3 - 21*A*a*tan(1/2*d*x + 1/2*c)^2 + 6*A*a*tan(1/2*d*x + 1/2*c) - A*a)/tan(1/2*d*x + 1/2*c)^3)/d
```



### 3.21 $\int \csc(c+dx)(a-a \csc(c+dx))(A-A \csc(c+dx)) dx$

**Optimal.** Leaf size=51

$$\frac{2aA \cot(c+dx)}{d} - \frac{3aA \tanh^{-1}(\cos(c+dx))}{2d} - \frac{aA \cot(c+dx) \csc(c+dx)}{2d}$$

[Out]  $(-3*a*A*ArcTanh[Cos[c + d*x]])/(2*d) + (2*a*A*Cot[c + d*x])/d - (a*A*Cot[c + d*x]*Csc[c + d*x])/(2*d)$

**Rubi [A]** time = 0.0554934, antiderivative size = 51, normalized size of antiderivative = 1., number of steps used = 6, number of rules used = 6, integrand size = 29,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.207$ , Rules used = {21, 3788, 3767, 8, 4046, 3770}

$$\frac{2aA \cot(c+dx)}{d} - \frac{3aA \tanh^{-1}(\cos(c+dx))}{2d} - \frac{aA \cot(c+dx) \csc(c+dx)}{2d}$$

Antiderivative was successfully verified.

[In]  $\text{Int}[Csc[c + d*x]*(a - a*Csc[c + d*x])*(A - A*Csc[c + d*x]),x]$

[Out]  $(-3*a*A*ArcTanh[Cos[c + d*x]])/(2*d) + (2*a*A*Cot[c + d*x])/d - (a*A*Cot[c + d*x]*Csc[c + d*x])/(2*d)$

#### Rule 21

$\text{Int}[(u_.)*((a_.) + (b_.)*(v_.))^{(m_.)*((c_.) + (d_.)*(v_.))^{(n_.)}, x\_Symbol] \rightarrow \text{Dist}[(b/d)^m, \text{Int}[u*(c + d*v)^{(m+n)}, x], x] /; \text{FreeQ}\{a, b, c, d, n\}, x] \&\& \text{EqQ}[b*c - a*d, 0] \&\& \text{IntegerQ}[m] \&\& (!\text{IntegerQ}[n] || \text{SimplerQ}[c + d*x, a + b*x])$

#### Rule 3788

$\text{Int}[(\csc[(e_.) + (f_.)*(x_.)]*(d_.))^{(n_.)*(\csc[(e_.) + (f_.)*(x_.)]*(b_.) + (a_.))^{(n_.)}, x\_Symbol] \rightarrow \text{Dist}[(2*a*b)/d, \text{Int}[(d*Csc[e + f*x])^{(n+1)}, x], x] + \text{Int}[(d*Csc[e + f*x])^n*(a^2 + b^2*Csc[e + f*x]^2), x] /; \text{FreeQ}\{a, b, d, e, f, n\}, x]$

#### Rule 3767

$\text{Int}[\csc[(c_.) + (d_.)*(x_.)]^{(n_.)}, x\_Symbol] \rightarrow -\text{Dist}[d^{(-1)}, \text{Subst}[\text{Int}[\text{ExpandIntegrand}[(1 + x^2)^{(n/2 - 1)}, x], x], \text{Cot}[c + d*x]], x] /; \text{FreeQ}\{c,$

d}, x] && IGtQ[n/2, 0]

### Rule 8

Int[a\_, x\_Symbol] := Simp[a\*x, x] /; FreeQ[a, x]

### Rule 4046

Int[(csc[(e\_.) + (f\_.)\*(x\_)]\*(b\_.))^(m\_.)\*(csc[(e\_.) + (f\_.)\*(x\_)]^2\*(C\_. + (A\_.)), x\_Symbol] := -Simp[(C\*Cot[e + f\*x]\*(b\*Csc[e + f\*x])^m)/(f\*(m + 1)), x] + Dist[(C\*m + A\*(m + 1))/(m + 1), Int[(b\*Csc[e + f\*x])^m, x], x] /; FreeQ[{b, e, f, A, C, m}, x] && NeQ[C\*m + A\*(m + 1), 0] && !LeQ[m, -1]

### Rule 3770

Int[csc[(c\_.) + (d\_.)\*(x\_)], x\_Symbol] := -Simp[ArcTanh[Cos[c + d\*x]]/d, x] /; FreeQ[{c, d}, x]

### Rubi steps

$$\begin{aligned} \int \csc(c + dx)(a - a \csc(c + dx))(A - A \csc(c + dx)) dx &= \frac{A \int \csc(c + dx)(a - a \csc(c + dx))^2 dx}{a} \\ &= \frac{A \int \csc(c + dx)(a^2 + a^2 \csc^2(c + dx)) dx}{a} - (2aA) \int \csc^2(c + dx) dx \\ &= -\frac{aA \cot(c + dx) \csc(c + dx)}{2d} + \frac{1}{2}(3aA) \int \csc(c + dx) dx + \frac{(2aA)^2}{2d} \\ &= -\frac{3aA \tanh^{-1}(\cos(c + dx))}{2d} + \frac{2aA \cot(c + dx)}{d} - \frac{aA \cot(c + dx)}{2d} \end{aligned}$$

**Mathematica [B]** time = 0.0349635, size = 137, normalized size = 2.69

$$\frac{2aA \cot(c + dx)}{d} - \frac{aA \csc^2\left(\frac{1}{2}(c + dx)\right)}{8d} + \frac{aA \sec^2\left(\frac{1}{2}(c + dx)\right)}{8d} + \frac{aA \log\left(\sin\left(\frac{c}{2} + \frac{dx}{2}\right)\right)}{d} + \frac{aA \log\left(\sin\left(\frac{1}{2}(c + dx)\right)\right)}{2d} - \frac{aA \log\left(\cos\left(\frac{c}{2} + \frac{dx}{2}\right)\right)}{d}$$

Antiderivative was successfully verified.

[In] Integrate[Csc[c + d\*x]\*(a - a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x]),x]

[Out] (2\*a\*A\*Cot[c + d\*x])/d - (a\*A\*Csc[(c + d\*x)/2]^2)/(8\*d) - (a\*A\*Log[Cos[c/2 + (d\*x)/2]])/d - (a\*A\*Log[Cos[(c + d\*x)/2]])/(2\*d) + (a\*A\*Log[Sin[c/2 + (d\*x)/2]])/d

$x)/2]])/d + (a*A*\text{Log}[\text{Sin}[(c + d*x)/2]])/(2*d) + (a*A*\text{Sec}[(c + d*x)/2]^2)/(8*d)$

**Maple [A]** time = 0.048, size = 57, normalized size = 1.1

$$\frac{3 A a \ln (\csc (d x+c)-\cot (d x+c))}{2 d}+2 \frac{A a \cot (d x+c)}{d}-\frac{A a \cot (d x+c) \csc (d x+c)}{2 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `int(csc(d*x+c)*(a-a*csc(d*x+c))*(A-A*csc(d*x+c)),x)`

[Out]  $3/2/d*A*a*\ln(\csc(d*x+c)-\cot(d*x+c))+2*a*A*\cot(d*x+c)/d-1/2*a*A*\cot(d*x+c)*\csc(d*x+c)/d$

**Maxima [A]** time = 1.00386, size = 108, normalized size = 2.12

$$A a \left( \frac{2 \cos (d x+c)}{\cos (d x+c)^2-1} - \log (\cos (d x+c)+1) + \log (\cos (d x+c)-1) \right) - 4 A a \log (\cot (d x+c)+\csc (d x+c)) + \frac{8 A a}{\tan (d x+c)}$$

$4 d$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate(csc(d*x+c)*(a-a*csc(d*x+c))*(A-A*csc(d*x+c)),x, algorithm="maxima")`

[Out]  $1/4*(A*a*(2*\cos(d*x + c)/(\cos(d*x + c)^2 - 1) - \log(\cos(d*x + c) + 1) + \log(\cos(d*x + c) - 1)) - 4*A*a*\log(\cot(d*x + c) + \csc(d*x + c)) + 8*A*a/\tan(d*x + c))/d$

**Fricas [B]** time = 0.496086, size = 274, normalized size = 5.37

$$\frac{8 A a \cos (d x+c) \sin (d x+c)-2 A a \cos (d x+c)+3\left(A a \cos (d x+c)^2-A a\right) \log \left(\frac{1}{2} \cos (d x+c)+\frac{1}{2}\right)-3\left(A a \cos (d x+c)\right)}{4\left(d \cos (d x+c)^2-d\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x, algorithm="fricas")

[Out]  $-1/4*(8*A*a*\cos(d*x + c)*\sin(d*x + c) - 2*A*a*\cos(d*x + c) + 3*(A*a*\cos(d*x + c)^2 - A*a)*\log(1/2*\cos(d*x + c) + 1/2) - 3*(A*a*\cos(d*x + c)^2 - A*a)*\log(-1/2*\cos(d*x + c) + 1/2))/(d*\cos(d*x + c)^2 - d)$

**Sympy [F]** time = 0., size = 0, normalized size = 0.

$$Aa \left( \int \csc(c + dx) dx + \int -2 \csc^2(c + dx) dx + \int \csc^3(c + dx) dx \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x)

[Out]  $A*a*(\text{Integral}(\csc(c + d*x), x) + \text{Integral}(-2*\csc(c + d*x)**2, x) + \text{Integral}(\csc(c + d*x)**3, x))$

**Giac [A]** time = 1.498, size = 126, normalized size = 2.47

$$\frac{Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^2 + 12 Aa \log\left(\left|\tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)\right|\right) - 8 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right) - \frac{18 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^2 - 8 Aa \tan\left(\frac{1}{2} dx + \frac{1}{2} c\right) + Aa}{\tan\left(\frac{1}{2} dx + \frac{1}{2} c\right)^2}}{8 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(csc(d\*x+c)\*(a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c)),x, algorithm="giac")

[Out]  $1/8*(A*a*\tan(1/2*d*x + 1/2*c)^2 + 12*A*a*\log(\text{abs}(\tan(1/2*d*x + 1/2*c))) - 8*A*a*\tan(1/2*d*x + 1/2*c) - (18*A*a*\tan(1/2*d*x + 1/2*c)^2 - 8*A*a*\tan(1/2*d*x + 1/2*c) + A*a)/\tan(1/2*d*x + 1/2*c)^2)/d$

### 3.22 $\int (a - a \csc(c + dx))(A - A \csc(c + dx)) \sin(c + dx) dx$

Optimal. Leaf size=33

$$\frac{aA \cos(c + dx)}{d} - \frac{aA \tanh^{-1}(\cos(c + dx))}{d} - 2aAx$$

[Out]  $-2*a*A*x - (a*A*ArcTanh[Cos[c + d*x]])/d - (a*A*Cos[c + d*x])/d$

**Rubi [A]** time = 0.0575666, antiderivative size = 33, normalized size of antiderivative = 1., number of steps used = 5, number of rules used = 5, integrand size = 29,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.172$ , Rules used = {21, 3788, 8, 4045, 3770}

$$\frac{aA \cos(c + dx)}{d} - \frac{aA \tanh^{-1}(\cos(c + dx))}{d} - 2aAx$$

Antiderivative was successfully verified.

[In]  $\text{Int}[(a - a*\text{Csc}[c + d*x])*(A - A*\text{Csc}[c + d*x])*Sin[c + d*x], x]$

[Out]  $-2*a*A*x - (a*A*ArcTanh[Cos[c + d*x]])/d - (a*A*Cos[c + d*x])/d$

#### Rule 21

$\text{Int}[(u_.)*((a_.) + (b_.)*(v_))^{(m_.)}*((c_.) + (d_.)*(v_))^{(n_.)}, x\_Symbol] \rightarrow \text{Dist}[(b/d)^m, \text{Int}[u*(c + d*v)^{(m+n)}, x], x] /; \text{FreeQ}\{a, b, c, d, n\}, x] \&\& \text{EqQ}[b*c - a*d, 0] \&\& \text{IntegerQ}[m] \&\& (!\text{IntegerQ}[n] \|\ \text{SimplerQ}[c + d*x, a + b*x])$

#### Rule 3788

$\text{Int}[(\text{csc}[(e_.) + (f_.)*(x_)]*(d_.)^{(n_.)}*(\text{csc}[(e_.) + (f_.)*(x_)]*(b_.) + (a_.)^2, x\_Symbol] \rightarrow \text{Dist}[(2*a*b)/d, \text{Int}[(d*\text{Csc}[e + f*x])^{(n+1)}, x], x] + \text{Int}[(d*\text{Csc}[e + f*x])^n*(a^2 + b^2*\text{Csc}[e + f*x]^2), x] /; \text{FreeQ}\{a, b, d, e, f, n\}, x]$

#### Rule 8

$\text{Int}[a_, x\_Symbol] \rightarrow \text{Simp}[a*x, x] /; \text{FreeQ}[a, x]$

#### Rule 4045

```
Int[(csc[(e_.) + (f_.)*(x_.)]*(b_.))^(m_.)*(csc[(e_.) + (f_.)*(x_.)]^2*(C_.)
+ (A_.)), x_Symbol] :> Simp[(A*Cot[e + f*x]*(b*Csc[e + f*x])^m)/(f*m), x] +
Dist[(C*m + A*(m + 1))/(b^2*m), Int[(b*Csc[e + f*x])^(m + 2), x], x] /; FreeQ[{b, e, f, A, C}, x] && NeQ[C*m + A*(m + 1), 0] && LeQ[m, -1]
```

### Rule 3770

```
Int[csc[(c_.) + (d_.)*(x_.)], x_Symbol] :> -Simp[ArcTanh[Cos[c + d*x]]/d, x]
/; FreeQ[{c, d}, x]
```

### Rubi steps

$$\begin{aligned} \int (a - a \csc(c + dx))(A - A \csc(c + dx)) \sin(c + dx) dx &= \frac{A \int (a - a \csc(c + dx))^2 \sin(c + dx) dx}{a} \\ &= \frac{A \int (a^2 + a^2 \csc^2(c + dx)) \sin(c + dx) dx}{a} - (2aA) \int 1 dx \\ &= -2aAx - \frac{aA \cos(c + dx)}{d} + (aA) \int \csc(c + dx) dx \\ &= -2aAx - \frac{aA \tanh^{-1}(\cos(c + dx))}{d} - \frac{aA \cos(c + dx)}{d} \end{aligned}$$

**Mathematica [B]** time = 0.0267257, size = 72, normalized size = 2.18

$$\frac{aA \sin(c) \sin(dx)}{d} - \frac{aA \cos(c) \cos(dx)}{d} + \frac{aA \log\left(\sin\left(\frac{c}{2} + \frac{dx}{2}\right)\right)}{d} - \frac{aA \log\left(\cos\left(\frac{c}{2} + \frac{dx}{2}\right)\right)}{d} - 2aAx$$

Antiderivative was successfully verified.

```
[In] Integrate[(a - a*Csc[c + d*x])*(A - A*Csc[c + d*x])*Sin[c + d*x], x]
```

```
[Out] -2*a*A*x - (a*A*Cos[c]*Cos[d*x])/d - (a*A*Log[Cos[c/2 + (d*x)/2])/d + (a*A*Log[Sin[c/2 + (d*x)/2])/d + (a*A*Sin[c]*Sin[d*x])/d
```

**Maple [A]** time = 0.041, size = 50, normalized size = 1.5

$$-2aAx - \frac{Aa \cos(dx + c)}{d} + \frac{Aa \ln(\csc(dx + c) - \cot(dx + c))}{d} - 2 \frac{Aac}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `int((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c),x)`

[Out]  $-2*a*A*x-a*A*\cos(d*x+c)/d+1/d*A*a*\ln(\csc(d*x+c)-\cot(d*x+c))-2/d*A*a*c$

**Maxima [A]** time = 0.994928, size = 55, normalized size = 1.67

$$\frac{2(dx+c)Aa + Aa \cos(dx+c) + Aa \log(\cot(dx+c) + \csc(dx+c))}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c),x, algorithm="maxima")`

[Out]  $-(2*(d*x + c)*A*a + A*a*\cos(d*x + c) + A*a*\log(\cot(d*x + c) + \csc(d*x + c)))/d$

**Fricas [A]** time = 0.502994, size = 149, normalized size = 4.52

$$\frac{4Aadx + 2Aa \cos(dx+c) + Aa \log\left(\frac{1}{2} \cos(dx+c) + \frac{1}{2}\right) - Aa \log\left(-\frac{1}{2} \cos(dx+c) + \frac{1}{2}\right)}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c),x, algorithm="fricas")`

[Out]  $-1/2*(4*A*a*d*x + 2*A*a*\cos(d*x + c) + A*a*\log(1/2*\cos(d*x + c) + 1/2) - A*a*\log(-1/2*\cos(d*x + c) + 1/2))/d$

**Sympy [A]** time = 12.3135, size = 82, normalized size = 2.48

$$-2Aax + Aa \left( \begin{array}{ll} \left( \frac{-\cot(c+dx)}{d \csc(c+dx)} \right) & \text{for } d \neq 0 \\ \frac{x}{\csc(c)} & \text{otherwise} \end{array} \right) + Aa \left( \begin{array}{ll} \left( \frac{x \cot(c) \csc(c)}{\cot(c)+\csc(c)} + \frac{x \csc^2(c)}{\cot(c)+\csc(c)} \right) & \text{for } d = 0 \\ -\frac{\log(\cot(c+dx)+\csc(c+dx))}{d} & \text{otherwise} \end{array} \right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c),x)
```

```
[Out] -2*A*a*x + A*a*Piecewise((-cot(c + d*x)/(d*csc(c + d*x)), Ne(d, 0)), (x/csc
(c), True)) + A*a*Piecewise((x*cot(c)*csc(c)/(cot(c) + csc(c)) + x*csc(c)**
2/(cot(c) + csc(c)), Eq(d, 0)), (-log(cot(c + d*x) + csc(c + d*x))/d, True)
)
```

**Giac [A]** time = 1.37157, size = 66, normalized size = 2.

$$\frac{2(dx+c)Aa - Aa \log\left(\left|\tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)\right|\right) + \frac{2Aa}{\tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^2 + 1}}{d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c),x, algorithm="giac")
```

```
[Out] -(2*(d*x + c)*A*a - A*a*log(abs(tan(1/2*d*x + 1/2*c)))) + 2*A*a/(tan(1/2*d*x
+ 1/2*c)^2 + 1))/d
```



### 3.23 $\int (a - a \csc(c + dx))(A - A \csc(c + dx)) \sin^2(c + dx) dx$

**Optimal.** Leaf size=42

$$\frac{2aA \cos(c + dx)}{d} - \frac{aA \sin(c + dx) \cos(c + dx)}{2d} + \frac{3aAx}{2}$$

[Out] (3\*a\*A\*x)/2 + (2\*a\*A\*Cos[c + d\*x])/d - (a\*A\*Cos[c + d\*x]\*Sin[c + d\*x])/(2\*d)

**Rubi [A]** time = 0.0687744, antiderivative size = 42, normalized size of antiderivative = 1., number of steps used = 5, number of rules used = 5, integrand size = 31,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.161$ , Rules used = {21, 3788, 2638, 4045, 8}

$$\frac{2aA \cos(c + dx)}{d} - \frac{aA \sin(c + dx) \cos(c + dx)}{2d} + \frac{3aAx}{2}$$

Antiderivative was successfully verified.

[In] Int[(a - a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x])\*Sin[c + d\*x]^2,x]

[Out] (3\*a\*A\*x)/2 + (2\*a\*A\*Cos[c + d\*x])/d - (a\*A\*Cos[c + d\*x]\*Sin[c + d\*x])/(2\*d)

#### Rule 21

```
Int[(u_)*((a_) + (b_)*(v_))^(m_)*((c_) + (d_)*(v_))^(n_), x_Symbol] :>
  Dist[(b/d)^m, Int[u*(c + d*v)^(m + n), x], x] /; FreeQ[{a, b, c, d, n}, x]
  && EqQ[b*c - a*d, 0] && IntegerQ[m] && (!IntegerQ[n] || SimplerQ[c + d*x,
  a + b*x])
```

#### Rule 3788

```
Int[(csc[(e_) + (f_)*(x_)]*(d_))^(n_)*(csc[(e_) + (f_)*(x_)]*(b_) +
(a_))^(2), x_Symbol] :> Dist[(2*a*b)/d, Int[(d*Csc[e + f*x])^(n + 1), x], x]
+ Int[(d*Csc[e + f*x])^n*(a^2 + b^2*Csc[e + f*x]^2), x] /; FreeQ[{a, b, d,
e, f, n}, x]
```

#### Rule 2638

```
Int[sin[(c_) + (d_)*(x_)], x_Symbol] :> -Simp[Cos[c + d*x]/d, x] /; FreeQ
[{c, d}, x]
```

Rule 4045

Int[(csc[(e\_.) + (f\_.)\*(x\_.)]\*(b\_.))^(m\_.)\*(csc[(e\_.) + (f\_.)\*(x\_.)]^2\*(C\_.) + (A\_.)), x\_Symbol] :> Simp[(A\*Cot[e + f\*x]\*(b\*Csc[e + f\*x])^m)/(f\*m), x] + Dist[(C\*m + A\*(m + 1))/(b^2\*m), Int[(b\*Csc[e + f\*x])^(m + 2), x], x] /; FreeQ[{b, e, f, A, C}, x] && NeQ[C\*m + A\*(m + 1), 0] && LeQ[m, -1]

Rule 8

Int[a\_, x\_Symbol] :> Simp[a\*x, x] /; FreeQ[a, x]

Rubi steps

$$\begin{aligned} \int (a - a \csc(c + dx))(A - A \csc(c + dx)) \sin^2(c + dx) dx &= \frac{A \int (a - a \csc(c + dx))^2 \sin^2(c + dx) dx}{a} \\ &= \frac{A \int (a^2 + a^2 \csc^2(c + dx)) \sin^2(c + dx) dx}{a} - (2aA) \int \sin(c + dx) dx \\ &= \frac{2aA \cos(c + dx)}{d} - \frac{aA \cos(c + dx) \sin(c + dx)}{2d} + \frac{1}{2}(3aA) \int 1 dx \\ &= \frac{3aAx}{2} + \frac{2aA \cos(c + dx)}{d} - \frac{aA \cos(c + dx) \sin(c + dx)}{2d} \end{aligned}$$

**Mathematica [A]** time = 0.0553226, size = 35, normalized size = 0.83

$$\frac{aA(6(c + dx) - \sin(2(c + dx)) + 8 \cos(c + dx))}{4d}$$

Antiderivative was successfully verified.

[In] Integrate[(a - a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x])\*Sin[c + d\*x]^2,x]

[Out] (a\*A\*(6\*(c + d\*x) + 8\*Cos[c + d\*x] - Sin[2\*(c + d\*x)]))/(4\*d)

**Maple [A]** time = 0.041, size = 49, normalized size = 1.2

$$\frac{1}{d} \left( Aa \left( -\frac{\cos(dx + c) \sin(dx + c)}{2} + \frac{dx}{2} + \frac{c}{2} \right) + 2Aa \cos(dx + c) + Aa(dx + c) \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `int((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c)^2,x)`

[Out] `1/d*(A*a*(-1/2*cos(d*x+c)*sin(d*x+c)+1/2*d*x+1/2*c)+2*A*a*cos(d*x+c)+A*a*(d*x+c))`

**Maxima [A]** time = 0.994249, size = 63, normalized size = 1.5

$$\frac{(2 dx + 2 c - \sin(2 dx + 2 c))Aa + 4 (dx + c)Aa + 8 Aa \cos(dx + c)}{4 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c)^2,x, algorithm="maxima")`

[Out] `1/4*((2*d*x + 2*c - sin(2*d*x + 2*c))*A*a + 4*(d*x + c)*A*a + 8*A*a*cos(d*x + c))/d`

**Fricas [A]** time = 0.472881, size = 97, normalized size = 2.31

$$\frac{3 A a d x - A a \cos(dx + c) \sin(dx + c) + 4 A a \cos(dx + c)}{2 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c)^2,x, algorithm="fricas")`

[Out] `1/2*(3*A*a*d*x - A*a*cos(d*x + c)*sin(d*x + c) + 4*A*a*cos(d*x + c))/d`

**Sympy [A]** time = 10.545, size = 105, normalized size = 2.5

$$\begin{cases} \frac{Aax \cot^2(c+dx)}{2 \csc^2(c+dx)} + Aax + \frac{Aax}{2 \csc^2(c+dx)} + \frac{2Aa \cot(c+dx)}{d \csc(c+dx)} - \frac{Aa \cot(c+dx)}{2d \csc^2(c+dx)} & \text{for } d \neq 0 \\ \frac{x(-A \csc(c)+A)(-a \csc(c)+a)}{\csc^2(c)} & \text{otherwise} \end{cases}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c)**2,x)
```

```
[Out] Piecewise((A*a*x*cot(c + d*x)**2/(2*csc(c + d*x)**2) + A*a*x + A*a*x/(2*csc
(c + d*x)**2) + 2*A*a*cot(c + d*x)/(d*csc(c + d*x)) - A*a*cot(c + d*x)/(2*d
*csc(c + d*x)**2), Ne(d, 0)), (x*(-A*csc(c) + A)*(-a*csc(c) + a)/csc(c)**2,
True))
```

**Giac [B]** time = 1.43383, size = 107, normalized size = 2.55

$$\frac{3(dx + c)Aa + \frac{2\left(Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^3 + 4Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^2 - Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right) + 4Aa\right)}{\left(\tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^2 + 1\right)^2}}{2d}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c)^2,x, algorithm="giac
")
```

```
[Out] 1/2*(3*(d*x + c)*A*a + 2*(A*a*tan(1/2*d*x + 1/2*c)^3 + 4*A*a*tan(1/2*d*x +
1/2*c)^2 - A*a*tan(1/2*d*x + 1/2*c) + 4*A*a)/(tan(1/2*d*x + 1/2*c)^2 + 1)^2
)/d
```

### 3.24 $\int (a - a \csc(c + dx))(A - A \csc(c + dx)) \sin^3(c + dx) dx$

**Optimal.** Leaf size=54

$$\frac{aA \cos^3(c + dx)}{3d} - \frac{2aA \cos(c + dx)}{d} + \frac{aA \sin(c + dx) \cos(c + dx)}{d} - aAx$$

[Out]  $-(aA*x) - (2*a*A*\text{Cos}[c + d*x])/d + (a*A*\text{Cos}[c + d*x]^3)/(3*d) + (a*A*\text{Cos}[c + d*x]*\text{Sin}[c + d*x])/d$

**Rubi [A]** time = 0.0902227, antiderivative size = 54, normalized size of antiderivative = 1., number of steps used = 7, number of rules used = 6, integrand size = 31,  $\frac{\text{number of rules}}{\text{integrand size}} = 0.194$ , Rules used = {21, 3788, 2635, 8, 4044, 3013}

$$\frac{aA \cos^3(c + dx)}{3d} - \frac{2aA \cos(c + dx)}{d} + \frac{aA \sin(c + dx) \cos(c + dx)}{d} - aAx$$

Antiderivative was successfully verified.

[In]  $\text{Int}[(a - a*\text{Csc}[c + d*x])*(A - A*\text{Csc}[c + d*x])*\text{Sin}[c + d*x]^3, x]$

[Out]  $-(aA*x) - (2*a*A*\text{Cos}[c + d*x])/d + (a*A*\text{Cos}[c + d*x]^3)/(3*d) + (a*A*\text{Cos}[c + d*x]*\text{Sin}[c + d*x])/d$

#### Rule 21

$\text{Int}[(u_*)*((a_*) + (b_*)*(v_*))^{(m_*)}*((c_*) + (d_*)*(v_*))^{(n_*)}, x\_Symbol] \rightarrow \text{Dist}[(b/d)^m, \text{Int}[u*(c + d*v)^{(m+n)}, x], x] /; \text{FreeQ}\{a, b, c, d, n\}, x] \&\& \text{EqQ}[b*c - a*d, 0] \&\& \text{IntegerQ}[m] \&\& (!\text{IntegerQ}[n] || \text{SimplerQ}[c + d*x, a + b*x])$

#### Rule 3788

$\text{Int}[(\text{csc}[e_*] + (f_*)*(x_*))*(d_*)^{(n_*)}*(\text{csc}[e_*] + (f_*)*(x_*))*(b_*) + (a_*)^2, x\_Symbol] \rightarrow \text{Dist}[(2*a*b)/d, \text{Int}[(d*\text{Csc}[e + f*x])^{(n+1)}, x], x] + \text{Int}[(d*\text{Csc}[e + f*x])^n*(a^2 + b^2*\text{Csc}[e + f*x]^2), x] /; \text{FreeQ}\{a, b, d, e, f, n\}, x]$

#### Rule 2635

$\text{Int}[(b_*)*\text{sin}[(c_*) + (d_*)*(x_*)]^{(n_*)}, x\_Symbol] \rightarrow -\text{Simp}[(b*\text{Cos}[c + d*x])*(b*\text{Sin}[c + d*x])^{(n-1)}]/(d*n), x] + \text{Dist}[(b^2*(n-1))/n, \text{Int}[(b*\text{Sin}[c$

+ d\*x])^(n - 2), x], x] /; FreeQ[{b, c, d}, x] && GtQ[n, 1] && IntegerQ[2\*n]

### Rule 8

Int[a\_, x\_Symbol] := Simp[a\*x, x] /; FreeQ[a, x]

### Rule 4044

Int[csc[(e\_.) + (f\_.)\*(x\_)]^(m\_.)\*(csc[(e\_.) + (f\_.)\*(x\_)]^2\*(C\_.) + (A\_.)),  
x\_Symbol] := Int[(C + A\*Sin[e + f\*x]^2)/Sin[e + f\*x]^(m + 2), x] /; FreeQ[  
{e, f, A, C}, x] && NeQ[C\*m + A\*(m + 1), 0] && ILtQ[(m + 1)/2, 0]

### Rule 3013

Int[sin[(e\_.) + (f\_.)\*(x\_)]^(m\_.)\*((A\_) + (C\_.)\*sin[(e\_.) + (f\_.)\*(x\_)]^2),  
x\_Symbol] := -Dist[f^(-1), Subst[Int[(1 - x^2)^(m - 1)/2]\*(A + C - C\*x^2)  
, x], x, Cos[e + f\*x]], x] /; FreeQ[{e, f, A, C}, x] && IGtQ[(m + 1)/2, 0]

### Rubi steps

$$\begin{aligned}
 \int (a - a \csc(c + dx))(A - A \csc(c + dx)) \sin^3(c + dx) dx &= \frac{A \int (a - a \csc(c + dx))^2 \sin^3(c + dx) dx}{a} \\
 &= \frac{A \int (a^2 + a^2 \csc^2(c + dx)) \sin^3(c + dx) dx}{a} - (2aA) \int \sin^2(c + dx) dx \\
 &= \frac{aA \cos(c + dx) \sin(c + dx)}{d} + \frac{A \int \sin(c + dx) (a^2 + a^2 \sin^2(c + dx)) dx}{a} \\
 &= -aAx + \frac{aA \cos(c + dx) \sin(c + dx)}{d} - \frac{A \text{Subst}\left(\int (2a^2 - a^2 x^2) dx\right)}{ad} \\
 &= -aAx - \frac{2aA \cos(c + dx)}{d} + \frac{aA \cos^3(c + dx)}{3d} + \frac{aA \cos(c + dx)}{d}
 \end{aligned}$$

**Mathematica [A]** time = 0.109858, size = 44, normalized size = 0.81

$$\frac{aA(6(\sin(2(c + dx)) - 2(c + dx)) - 21 \cos(c + dx) + \cos(3(c + dx)))}{12d}$$

Antiderivative was successfully verified.

[In] Integrate[(a - a\*Csc[c + d\*x])\*(A - A\*Csc[c + d\*x])\*Sin[c + d\*x]^3,x]

[Out] (a\*A\*(-21\*Cos[c + d\*x] + Cos[3\*(c + d\*x)] + 6\*(-2\*(c + d\*x) + Sin[2\*(c + d\*x)])))/(12\*d)

**Maple [A]** time = 0.048, size = 62, normalized size = 1.2

$$\frac{1}{d} \left( -\frac{Aa \left( 2 + (\sin(dx + c))^2 \right) \cos(dx + c)}{3} - 2Aa \left( -\frac{1}{2} \cos(dx + c) \sin(dx + c) + \frac{1}{2} dx + \frac{c}{2} \right) - Aa \cos(dx + c) \right)$$

Verification of antiderivative is not currently implemented for this CAS.

[In] int((a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c))/csc(d\*x+c)^3,x)

[Out] 1/d\*(-1/3\*A\*a\*(2+sin(d\*x+c)^2)\*cos(d\*x+c)-2\*A\*a\*(-1/2\*cos(d\*x+c)\*sin(d\*x+c)+1/2\*d\*x+1/2\*c)-A\*a\*cos(d\*x+c))

**Maxima [A]** time = 1.00711, size = 81, normalized size = 1.5

$$\frac{2 \left( \cos(dx + c)^3 - 3 \cos(dx + c) \right) Aa - 3 (2 dx + 2 c - \sin(2 dx + 2 c)) Aa - 6 Aa \cos(dx + c)}{6 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c))/csc(d\*x+c)^3,x, algorithm="maxima")

[Out] 1/6\*(2\*(cos(d\*x + c)^3 - 3\*cos(d\*x + c))\*A\*a - 3\*(2\*d\*x + 2\*c - sin(2\*d\*x + 2\*c))\*A\*a - 6\*A\*a\*cos(d\*x + c))/d

**Fricas [A]** time = 0.481526, size = 128, normalized size = 2.37

$$\frac{Aa \cos(dx + c)^3 - 3 Aa dx + 3 Aa \cos(dx + c) \sin(dx + c) - 6 Aa \cos(dx + c)}{3 d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate((a-a\*csc(d\*x+c))\*(A-A\*csc(d\*x+c))/csc(d\*x+c)^3,x, algorithm="fricas")

[Out]  $\frac{1}{3}*(A*a*\cos(d*x + c)^3 - 3*A*a*d*x + 3*A*a*\cos(d*x + c)*\sin(d*x + c) - 6*A*a*\cos(d*x + c))/d$

**Sympy [A]** time = 13.726, size = 139, normalized size = 2.57

$$\begin{cases} -\frac{Aax \cot^2(c+dx)}{\csc^2(c+dx)} - \frac{Aax}{\csc^2(c+dx)} - \frac{2Aa \cot^3(c+dx)}{3d \csc^3(c+dx)} - \frac{Aa \cot(c+dx)}{d \csc(c+dx)} + \frac{Aa \cot(c+dx)}{d \csc^2(c+dx)} - \frac{Aa \cot(c+dx)}{d \csc^3(c+dx)} & \text{for } d \neq 0 \\ \frac{x(-A \csc(c)+A)(-a \csc(c)+a)}{\csc^3(c)} & \text{otherwise} \end{cases}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c)**3,x)`

[Out] `Piecewise((-A*a*x*cot(c + d*x)**2/csc(c + d*x)**2 - A*a*x/csc(c + d*x)**2 - 2*A*a*cot(c + d*x)**3/(3*d*csc(c + d*x)**3) - A*a*cot(c + d*x)/(d*csc(c + d*x)) + A*a*cot(c + d*x)/(d*csc(c + d*x)**2) - A*a*cot(c + d*x)/(d*csc(c + d*x)**3), Ne(d, 0)), (x*(-A*csc(c) + A)*(-a*csc(c) + a)/csc(c)**3, True))`

**Giac [A]** time = 1.52714, size = 128, normalized size = 2.37

$$\frac{3(dx+c)Aa + \frac{2\left(3Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^5 + 3Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^4 + 12Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^2 - 3Aa \tan\left(\frac{1}{2}dx + \frac{1}{2}c\right) + 5Aa\right)}{\left(\tan\left(\frac{1}{2}dx + \frac{1}{2}c\right)^2 + 1\right)^3}{3d}$$

Verification of antiderivative is not currently implemented for this CAS.

[In] `integrate((a-a*csc(d*x+c))*(A-A*csc(d*x+c))/csc(d*x+c)^3,x, algorithm="giac")`

[Out]  $-\frac{1}{3}*(3*(d*x + c)*A*a + 2*(3*A*a*\tan(1/2*d*x + 1/2*c)^5 + 3*A*a*\tan(1/2*d*x + 1/2*c)^4 + 12*A*a*\tan(1/2*d*x + 1/2*c)^2 - 3*A*a*\tan(1/2*d*x + 1/2*c) + 5*A*a)/(\tan(1/2*d*x + 1/2*c)^2 + 1)^3)/d$



# Chapter 4

## Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

### 4.0.1 Mathematica and Rubi grading function

```
1 (* Original version thanks to Albert Rich emailed on 03/21/2017 *)
2 (* ::Package:: *)
3
4 (* ::Subsection:: *)
5 (*GradeAntiderivative[result,optimal]*)
6
7
8 (* ::Text:: *)
9 (*If result and optimal are mathematical expressions, *)
10 (*      GradeAntiderivative[result,optimal] returns*)
11 (* "F" if the result fails to integrate an expression that*)
12 (*      is integrable*)
13 (* "C" if result involves higher level functions than necessary*)
14 (* "B" if result is more than twice the size of the optimal*)
15 (*      antiderivative*)
16 (* "A" if result can be considered optimal*)
17
18
19 GradeAntiderivative[result_,optimal_] :=
20   If[ExpnType[result]<=ExpnType[optimal],
21     If[FreeQ[result,Complex] || Not[FreeQ[optimal,Complex]],
```

```

22     If[LeafCount[result]<=2*LeafCount[optimal],
23         "A",
24         "B"],
25     "C"],
26 If[FreeQ[result,Integrate] && FreeQ[result,Int],
27     "C",
28     "F"]]
29
30
31 (* ::Text:: *)
32 (*The following summarizes the type number assigned an *)
33 (*expression based on the functions it involves*)
34 (*1 = rational function*)
35 (*2 = algebraic function*)
36 (*3 = elementary function*)
37 (*4 = special function*)
38 (*5 = hyperpergeometric function*)
39 (*6 = appell function*)
40 (*7 = rootsum function*)
41 (*8 = integrate function*)
42 (*9 = unknown function*)
43
44
45 ExpnType[expn_] :=
46     If[AtomQ[expn],
47         1,
48     If[ListQ[expn],
49         Max[Map[ExpnType,expn]],
50     If[Head[expn]===Power,
51         If[IntegerQ[expn[[2]]],
52             ExpnType[expn[[1]]],
53         If[Head[expn[[2]]]===Rational,
54             If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
55                 1,
56                 Max[ExpnType[expn[[1]],2]],
57             Max[ExpnType[expn[[1]],ExpnType[expn[[2]],3]],
58     If[Head[expn]===Plus || Head[expn]===Times,
59         Max[ExpnType[First[expn]],ExpnType[Rest[expn]]],
60     If[ElementaryFunctionQ[Head[expn]],
61         Max[3,ExpnType[expn[[1]]]],
62     If[SpecialFunctionQ[Head[expn]],
63         Apply[Max,Append[Map[ExpnType,Apply[List,expn]],4]],
64     If[HypergeometricFunctionQ[Head[expn]],
65         Apply[Max,Append[Map[ExpnType,Apply[List,expn]],5]],
66     If[AppellFunctionQ[Head[expn]],
67         Apply[Max,Append[Map[ExpnType,Apply[List,expn]],6]],
68     If[Head[expn]===RootSum,
```

```

69   Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 7]],
70   If[Head[expn]===Integrate || Head[expn]===Int,
71     Apply[Max, Append[Map[ExpnType, Apply[List, expn]], 8]],
72   9]]]]]]]]]]
73
74
75 ElementaryFunctionQ[func_] :=
76   MemberQ[{
77     Exp, Log,
78     Sin, Cos, Tan, Cot, Sec, Csc,
79     ArcSin, ArcCos, ArcTan, ArcCot, ArcSec, ArcCsc,
80     Sinh, Cosh, Tanh, Coth, Sech, Csch,
81     ArcSinh, ArcCosh, ArcTanh, ArcCoth, ArcSech, ArcCsch
82   }, func]
83
84
85 SpecialFunctionQ[func_] :=
86   MemberQ[{
87     Erf, Erfc, Erfi,
88     FresnelS, FresnelC,
89     ExpIntegralE, ExpIntegralEi, LogIntegral,
90     SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
91     Gamma, LogGamma, PolyGamma,
92     Zeta, PolyLog, ProductLog,
93     EllipticF, EllipticE, EllipticPi
94   }, func]
95
96
97 HypergeometricFunctionQ[func_] :=
98   MemberQ[{Hypergeometric1F1, Hypergeometric2F1, HypergeometricPFQ}, func]
99
100
101 AppellFunctionQ[func_] :=
102   MemberQ[{AppellF1}, func]

```

## 4.0.2 Maple grading function

```

1 # File: GradeAntiderivative.mpl
2 # Original version thanks to Albert Rich emailed on 03/21/2017
3
4 #Nasser 03/22/2017 Use Maple leaf count instead since buildin
5 #Nasser 03/23/2017 missing 'ln' for ElementaryFunctionQ added
6 #Nasser 03/24/2017 corrected the check for complex result
7 #Nasser 10/27/2017 check for leafsize and do not call ExpnType()
8 #
9 #Nasser 12/22/2019 Added debug flag, added 'dilog' to special functions
10 #
11 see problem 156, file Apostol_Problems

```

```

11
12 GradeAntiderivative := proc(result,optimal)
13 local leaf_count_result, leaf_count_optimal,ExpnType_result,ExpnType_optimal,
    debug:=false;
14
15     leaf_count_result:=leafcount(result);
16     #do NOT call ExpnType() if leaf size is too large. Recursion problem
17     if leaf_count_result > 500000 then
18         return "B";
19     fi;
20
21     leaf_count_optimal:=leafcount(optimal);
22
23     ExpnType_result:=ExpnType(result);
24     ExpnType_optimal:=ExpnType(optimal);
25
26     if debug then
27         print("ExpnType_result",ExpnType_result," ExpnType_optimal=",
    ExpnType_optimal);
28     fi;
29
30 # If result and optimal are mathematical expressions,
31 # GradeAntiderivative[result,optimal] returns
32 #   "F" if the result fails to integrate an expression that
33 #     is integrable
34 #   "C" if result involves higher level functions than necessary
35 #   "B" if result is more than twice the size of the optimal
36 #     antiderivative
37 #   "A" if result can be considered optimal
38
39 #This check below actually is not needed, since I only
40 #call this grading only for passed integrals. i.e. I check
41 #for "F" before calling this. But no harm of keeping it here.
42 #just in case.
43
44
45 if not type(result,freeof('int')) then
46     return "F";
47 end if;
48
49
50 if ExpnType_result<=ExpnType_optimal then
51     if debug then
52         print("ExpnType_result<=ExpnType_optimal");
53     fi;
54     if is_contains_complex(result) then
55         if is_contains_complex(optimal) then

```

```

56     if debug then
57         print("both result and optimal complex");
58     fi;
59     #both result and optimal complex
60     if leaf_count_result<=2*leaf_count_optimal then
61         return "A";
62     else
63         return "B";
64     end if
65     else #result contains complex but optimal is not
66         if debug then
67             print("result contains complex but optimal is not");
68         fi;
69         return "C";
70     end if
71     else # result do not contain complex
72         # this assumes optimal do not as well
73         if debug then
74             print("result do not contain complex, this assumes optimal do not
as well");
75         fi;
76         if leaf_count_result<=2*leaf_count_optimal then
77             if debug then
78                 print("leaf_count_result<=2*leaf_count_optimal");
79             fi;
80             return "A";
81         else
82             if debug then
83                 print("leaf_count_result>2*leaf_count_optimal");
84             fi;
85             return "B";
86         end if
87     end if
88     else #ExpnType(result) > ExpnType(optimal)
89         if debug then
90             print("ExpnType(result) > ExpnType(optimal)");
91         fi;
92         return "C";
93     end if
94
95 end proc:
96
97 #
98 # is_contains_complex(result)
99 # takes expressions and returns true if it contains "I" else false
100 #
101 #Nasser 032417

```

```

102 is_contains_complex:= proc(expression)
103   return (has(expression,I));
104 end proc:
105
106 # The following summarizes the type number assigned an expression
107 # based on the functions it involves
108 # 1 = rational function
109 # 2 = algebraic function
110 # 3 = elementary function
111 # 4 = special function
112 # 5 = hyperpergeometric function
113 # 6 = appell function
114 # 7 = rootsum function
115 # 8 = integrate function
116 # 9 = unknown function
117
118 ExpnType := proc(expn)
119   if type(expn,'atomic') then
120     1
121   elif type(expn,'list') then
122     apply(max,map(ExpnType,expn))
123   elif type(expn,'sqrt') then
124     if type(op(1,expn),'rational') then
125       1
126     else
127       max(2,ExpnType(op(1,expn)))
128     end if
129   elif type(expn,'^^') then
130     if type(op(2,expn),'integer') then
131       ExpnType(op(1,expn))
132     elif type(op(2,expn),'rational') then
133       if type(op(1,expn),'rational') then
134         1
135       else
136         max(2,ExpnType(op(1,expn)))
137       end if
138     else
139       max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
140     end if
141   elif type(expn,'^+^') or type(expn,'`*`') then
142     max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
143   elif ElementaryFunctionQ(op(0,expn)) then
144     max(3,ExpnType(op(1,expn)))
145   elif SpecialFunctionQ(op(0,expn)) then
146     max(4,apply(max,map(ExpnType,[op(expn)])))
147   elif HypergeometricFunctionQ(op(0,expn)) then
148     max(5,apply(max,map(ExpnType,[op(expn)])))

```

```

149   elif AppellFunctionQ(op(0,expn)) then
150       max(6,apply(max,map(ExpnType,[op(expn)])))
151   elif op(0,expn)='int' then
152       max(8,apply(max,map(ExpnType,[op(expn)]))) else
153       9
154   end if
155 end proc:
156
157
158 ElementaryFunctionQ := proc(func)
159     member(func,[
160         exp,log,ln,
161         sin,cos,tan,cot,sec,csc,
162         arcsin,arccos,arctan,arccot,arcsec,arccsc,
163         sinh,cosh,tanh,coth,sech,csch,
164         arcsinh,arccosh,arctanh,arccoth,arcsech,arccsch])
165 end proc:
166
167 SpecialFunctionQ := proc(func)
168     member(func,[
169         erf,erfc,erfi,
170         FresnelS,FresnelC,
171         Ei,Ei,Li,Si,Ci,Shi,Chi,
172         GAMMA,lnGAMMA,Psi,Zeta,polylog,dilog,LambertW,
173         EllipticF,EllipticE,EllipticPi])
174 end proc:
175
176 HypergeometricFunctionQ := proc(func)
177     member(func,[Hypergeometric1F1,hypergeom,HypergeometricPFQ])
178 end proc:
179
180 AppellFunctionQ := proc(func)
181     member(func,[AppellF1])
182 end proc:
183
184 # u is a sum or product. rest(u) returns all but the
185 # first term or factor of u.
186 rest := proc(u) local v;
187     if nops(u)=2 then
188         op(2,u)
189     else
190         apply(op(0,u),op(2..nops(u),u))
191     end if
192 end proc:
193
194 #leafcount(u) returns the number of nodes in u.
195 #Nasser 3/23/17 Replaced by build-in leafCount from package in Maple

```

```

196 leafcount := proc(u)
197     MmaTranslator[Mma][LeafCount](u);
198 end proc:

```

### 4.0.3 Sympy grading function

```

1 #Dec 24, 2019. Nasser M. Abbasi:
2 #           Port of original Maple grading function by
3 #           Albert Rich to use with Sympy/Python
4 #Dec 27, 2019 Nasser. Added `RootSum`. See problem 177, Timofeev file
5 #           added 'exp_polar'
6 from sympy import *
7
8 def leaf_count(expr):
9     #sympy do not have leaf count function. This is approximation
10    return round(1.7*count_ops(expr))
11
12 def is_sqrt(expr):
13     if isinstance(expr,Pow):
14         if expr.args[1] == Rational(1,2):
15             return True
16         else:
17             return False
18     else:
19         return False
20
21 def is_elementary_function(func):
22     return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
23                    asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
24                    asinh,acosh,atanh,acoth,asech,acsch
25                    ]
26
27 def is_special_function(func):
28     return func in [ erf,erfc,erfi,
29                    fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
30                    gamma,loggamma,digamma,zeta,polylog,LambertW,
31                    elliptic_f,elliptic_e,elliptic_pi,exp_polar
32                    ]
33
34 def is_hypergeometric_function(func):
35     return func in [hyper]
36
37 def is_appell_function(func):
38     return func in [appellf1]
39
40 def is_atom(expn):
41     try:

```



```

42     if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
43         return True
44     else:
45         return False
46
47     except AttributeError as error:
48         return False
49
50 def expnType(expn):
51     debug=False
52     if debug:
53         print("expn=",expn,"type(expn)=",type(expn))
54
55     if is_atom(expn):
56         return 1
57     elif isinstance(expn,list):
58         return max(map(expnType, expn)) #apply(max,map(ExpnType,expn))
59     elif is_sqrt(expn):
60         if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
61             return 1
62         else:
63             return max(2,expnType(expn.args[0])) #max(2,ExpnType(op(1,expn)))
64     elif isinstance(expn,Pow): #type(expn,'^^')
65         if isinstance(expn.args[1],Integer): #type(op(2,expn),'integer')
66             return expnType(expn.args[0]) #ExpnType(op(1,expn))
67         elif isinstance(expn.args[1],Rational): #type(op(2,expn),'rational')
68             if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
69                 return 1
70             else:
71                 return max(2,expnType(expn.args[0])) #max(2,ExpnType(op(1,expn)
72 ))
73     else:
74         return max(3,expnType(expn.args[0]),expnType(expn.args[1])) #max(3,
75 ExpnType(op(1,expn)),ExpnType(op(2,expn)))
76     elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'+' or type
77 (expn,'*')
78         m1 = expnType(expn.args[0])
79         m2 = expnType(list(expn.args[1:]))
80         return max(m1,m2) #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
81     elif is_elementary_function(expn.func): #ElementaryFunctionQ(op(0,expn))
82         return max(3,expnType(expn.args[0])) #max(3,ExpnType(op(1,expn)))
83     elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
84         m1 = max(map(expnType, list(expn.args)))
85         return max(4,m1) #max(4,apply(max,map(ExpnType,[op(expn)])))
86     elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,
87 expn))
88         m1 = max(map(expnType, list(expn.args)))

```

```

85     return max(5,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
86 elif is_appell_function(expn.func):
87     m1 = max(map(expnType, list(expn.args)))
88     return max(6,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
89 elif isinstance(expn,RootSum):
90     m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType,
Apply[List,expn]],7]],
91     return max(7,m1)
92 elif str(expn).find("Integral") != -1:
93     m1 = max(map(expnType, list(expn.args)))
94     return max(8,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
95 else:
96     return 9
97
98 #main function
99 def grade_antiderivative(result,optimal):
100
101     leaf_count_result  = leaf_count(result)
102     leaf_count_optimal = leaf_count(optimal)
103
104     expnType_result  = expnType(result)
105     expnType_optimal = expnType(optimal)
106
107     if str(result).find("Integral") != -1:
108         return "F"
109
110     if expnType_result <= expnType_optimal:
111         if result.has(I):
112             if optimal.has(I): #both result and optimal complex
113                 if leaf_count_result <= 2*leaf_count_optimal:
114                     return "A"
115                 else:
116                     return "B"
117             else: #result contains complex but optimal is not
118                 return "C"
119         else: # result do not contain complex, this assumes optimal do not as
well
120             if leaf_count_result <= 2*leaf_count_optimal:
121                 return "A"
122             else:
123                 return "B"
124     else:
125         return "C"

```

## 4.0.4 SageMath grading function

1 #Dec 24, 2019. Nasser: Ported original Maple grading function by

```

2 #           Albert Rich to use with Sagemath. This is used to
3 #           grade Fracas, Giac and Maxima results.
4 #Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
5 #           'arctan2','floor','abs','log_integral'
6
7 from sage.all import *
8 from sage.symbolic.operators import add_vararg, mul_vararg
9
10 def tree(expr):
11     debug=False;
12     if debug:
13         print ("Enter tree(expr), expr=",expr)
14         print ("expr.operator()=",expr.operator())
15         print ("expr.operands()=",expr.operands())
16         print ("map(tree, expr.operands()=",map(tree, expr.operands()))
17
18     if expr.operator() is None:
19         return expr
20     else:
21         return [expr.operator()+list(map(tree, expr.operands()))
22
23 def leaf_count(anti):
24     debug=False;
25
26     if debug: print ("Enter leaf_count, anti=", anti, " len(anti)=", len(anti))
27
28     if len(anti) == 0: #special check for optimal being 0 for some test cases.
29         if debug: print ("len(anti) == 0")
30         return 1
31     else:
32         if debug: print ("round(1.35*len(flatten(tree(anti))))=",round(1.35*len(
33         flatten(tree(anti))))
34         return round(1.35*len(flatten(tree(anti)))) #fudge factor
35         #since this estimate of leaf count is bit lower than
36         #what it should be compared to Mathematica's
37
38 def is_sqrt(expr):
39     debug=False;
40     if expr.operator() == operator.pow: #isinstance(expr,Pow):
41         if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
42             if debug: print ("expr is sqrt")
43             return True
44         else:
45             return False
46     else:
47         return False

```

```

48 def is_elementary_function(func):
49     debug = False
50
51     m = func.name() in ['exp','log','ln',
52         'sin','cos','tan','cot','sec','csc',
53         'arcsin','arccos','arctan','arccot','arcsec','arccsc',
54         'sinh','cosh','tanh','coth','sech','csch',
55         'arcsinh','arccosh','arctanh','arccoth','arcsech','arccsch','sgn',
56         'arctan2','floor','abs'
57     ]
58     if debug:
59         if m:
60             print ("func ", func , " is elementary_function")
61         else:
62             print ("func ", func , " is NOT elementary_function")
63
64
65     return m
66
67 def is_special_function(func):
68     debug = False
69
70     if debug: print ("type(func)=", type(func))
71
72     m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
73         'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','
74     sinh_integral'
75         'Chi','cosh_integral','gamma','log_gamma','psi,zeta',
76         'polylog','lambert_w','elliptic_f','elliptic_e',
77         'elliptic_pi','exp_integral_e','log_integral']
78
79     if debug:
80         print ("m=",m)
81         if m:
82             print ("func ", func ," is special_function")
83         else:
84             print ("func ", func ," is NOT special_function")
85
86     return m
87
88
89 def is_hypergeometric_function(func):
90     return func.name() in ['hypergeometric','hypergeometric_M','hypergeometric_U
91     ']
92
93 def is_appell_function(func):

```

```

93     return func.name() in ['hypergeometric']    #[appellf1] can't find this in
          sagemath
94
95 def is_atom(expn):
96
97     #thanks to answer at https://ask.sagemath.org/question/49179/what-is-sagemath-equivalent-to-atomic-type-in-maple/
98     try:
99         if expn.parent() is SR:
100             return expn.operator() is None
101         if expn.parent() in (ZZ, QQ, AA, QQbar):
102             return expn in expn.parent() # Should always return True
103         if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens"):
104             return expn in expn.parent().base_ring() or expn in expn.parent().
          gens()
105         return False
106
107     except AttributeError as error:
108         return False
109
110
111 def expnType(expn):
112     debug=False
113
114     if debug:
115         print(">>>>Enter expnType, expn=", expn)
116         print(">>>>is_atom(expn)=", is_atom(expn))
117
118     if is_atom(expn):
119         return 1
120     elif type(expn)==list:    #isinstance(expn,list):
121         return max(map(expnType, expn))    #apply(max,map(ExpnType,expn))
122     elif is_sqrt(expn):
123         if type(expn.operands()[0])==Rational: #type(isinstance(expn.args[0],
          Rational):
124             return 1
125         else:
126             return max(2,expnType(expn.operands()[0]))    #max(2,expnType(expn.
          args[0]))
127     elif expn.operator() == operator.pow:    #isinstance(expn,Pow)
128         if type(expn.operands()[1])==Integer:    #isinstance(expn.args[1],Integer)
129             return expnType(expn.operands()[0])    #expnType(expn.args[0])
130         elif type(expn.operands()[1])==Rational:    #isinstance(expn.args[1],
          Rational)
131             if type(expn.operands()[0])==Rational: #isinstance(expn.args[0],
          Rational)
132                 return 1

```

```

133         else:
134             return max(2,expnType(expn.operands()[0])) #max(2,expnType(expn.
args[0]))
135         else:
136             return max(3,expnType(expn.operands()[0]),expnType(expn.operands()
[1])) #max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1]))
137         elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #
isinstance(expn,Add) or isinstance(expn,Mul)
138             m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
139             m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
140             return max(m1,m2) #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
141         elif is_elementary_function(expn.operator()): #is_elementary_function(expn.
func)
142             return max(3,expnType(expn.operands()[0]))
143         elif is_special_function(expn.operator()): #is_special_function(expn.func)
144             m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(
expn.args)))
145             return max(4,m1) #max(4,m1)
146         elif is_hypergeometric_function(expn.operator()): #
is_hypergeometric_function(expn.func)
147             m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(
expn.args)))
148             return max(5,m1) #max(5,m1)
149         elif is_appell_function(expn.operator()):
150             m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(
expn.args)))
151             return max(6,m1) #max(6,m1)
152         elif str(expn).find("Integral") != -1: #this will never happen, since it
153             #is checked before calling the grading function that is passed.
154             #but kept it here.
155             m1 = max(map(expnType, expn.operands())) #max(map(expnType, list(
expn.args)))
156             return max(8,m1) #max(5,apply(max,map(ExpnType,[op(expn)])))
157         else:
158             return 9
159
160 #main function
161 def grade_antiderivative(result,optimal):
162     debug = False;
163
164     if debug: print ("Enter grade_antiderivative for sagemath")
165
166     leaf_count_result = leaf_count(result)
167     leaf_count_optimal = leaf_count(optimal)
168
169     if debug: print ("leaf_count_result=", leaf_count_result, "
leaf_count_optimal=",leaf_count_optimal)

```

```
170
171
172     expnType_result = expnType(result)
173     expnType_optimal = expnType(optimal)
174
175     if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",
176                     expnType_optimal)
177
178     if expnType_result <= expnType_optimal:
179         if result.has(I):
180             if optimal.has(I): #both result and optimal complex
181                 if leaf_count_result <= 2*leaf_count_optimal:
182                     return "A"
183             else:
184                 return "B"
185         else: #result contains complex but optimal is not
186             return "C"
187     else: # result do not contain complex, this assumes optimal do not as
188         well
189         if leaf_count_result <= 2*leaf_count_optimal:
190             return "A"
191         else:
192             return "B"
193     else:
194         return "C"
```